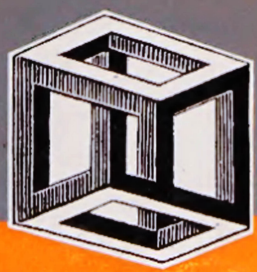
D. S. Mitrinović

Elementary Matrices



Tutorial Texts

D. S. MITRINOVIĆ • ELEMENTARY MATRICES

TUTORIAL TEXT No. 3

ELEMENTARY MATRICES

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IN COOPERATION WITH

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Adelaide, Australia

1965

P. NOORDHOFF LTD - GRONINGEN

THE NETHERLANDS

TUTORIAL TEXTS

NO 1 ELEMENTARY INEQUALITIES, 1964

NO 2 FUNCTIONS OF A COMPLEX VARIABLE, 1965

NO 3 ELEMENTARY MATRICES, 1965

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Printed in The Netherlands.

FOREWORD

The problems contained in this series have been collected over many years with the aim of providing students and teachers with material, the search for which would otherwise occupy much valuable time. Hitherto this concentrated material has only been accessible to the very restricted public able to read Serbian*.

I greatly welcome the appearance of the Tutorial Texts based on my problem collection. Cooperation with colleagues in Australia and the U.S.A. and their initiative have made it possible. For the preparation of this Text, I wish to thank R. B. Potts.

D. S. MITRINOVIĆ.

Belgrade, 23.III.1964.

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* Zbornik matematičkih problema, Belgrade, Vol. I (1962), Vol. II (1960) Vol. III (1960), published in cooperation with D. Adamović, V. Devidé, D. Djoković, D. Mihailović, Z. Pop-Stojanović, S. Prešic, J. Ulčar.

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INTRODUCTION

This tutorial text and problem collection is designed to introduce the student, at undergraduate or senior high school level, to the elementary properties of matrices and determinants. A knowledge of elementary algebra provides a sufficient background for almost all the material. It is hoped that, as a collection of problems, the book will prove a useful adjunct to regular course work. The introduction provides a concise resume of elementary matrix theory and the numerous problems should give students an opportunity for extensive practice in working examples. For ease of reference, the text and problems are classified into complementary sections.

The following selected references are intended as a guide to the very extensive literature on matrices.

1. For a brief survey of general matrix theory, see C. C. MacDuffee: *The Theory of Matrices*, Chelsea, New York, 1946.
2. An exhaustive treatment of determinants is given by T. Muir: *A Treatise on the Theory of Determinants*, Dover, New York, 1960.
3. An algebraic approach to the theory of matrices is given by H. Schwerdtfeger: *Introduction to Linear Algebra and the Theory of Matrices*, Noordhoff, Holland, 1950.
4. For a translation of an excellent Russian text, see F. R. Gantmacher: *The Theory of Matrices*, Vols. 1, 2. (Translated by K. A. Hirsch), Chelsea, New York, 1959.
5. A modern treatment, with extensive references to recent literature, is found in R. Bellman: *Introduction to Matrix Analysis*, McGraw-Hill, New York, 1960.
6. For applications of matrix theory in numerical analysis, see V. N. Faddeeva: *Computational Methods of Linear Algebra*, (translated by C. D. Benster), Dover, New York, 1959.
7. An inexpensive pamphlet listing the most important results of matrix theory is M. Marcus: *Basic Theorems in Matrix Theory*, National Bureau of Standards, Applied Mathematics Series No. 57, 1960.

§ 0.1 Elements of Matrix Calculus

0.1.1 DEFINITION. *A matrix A of order $m \times n$ is a rectangular array of mn elements arranged in m rows and n columns*:

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ . & . & \dots & \\ . & . & \dots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \tag{1}$$

where a_{ij} denotes the element in the ith *row and* jth *column and* $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$.

If $m = n$, A is called a (square) *matrix of order* n, or an n-*square matrix*.

A matrix of order $m \times 1$ is called a *column*.

A matrix of order $1 \times n$ is called a *row*.

0.1.2 RULES OF RECKONING

It will be assumed that the matrix elements belong to a field F (e.g. the field of complex numbers); the rules of reckoning for the matrices will be defined in terms of those assumed operating in F.

0.1.3 EQUALITY. Two matrices $A = [a_{ij}]$, and $B = [b_{ij}]$ are equal if and only if $a_{ij} = b_{ij}$ for all i, j (implying A and B have the same order).

0.1.4 ADDITION. The sum of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is defined to be

$$A + B = [a_{ij} + b_{ij}], \tag{2}$$

where it is implied that A and B have the same order.

Addition of matrices satisfies the same laws as addition in F, such as:

$$\textit{associative law}: \quad A+(B+C) = (A+B)+C, \tag{3}$$

$$\textit{commutative law}: \quad A+B = B+A. \tag{4}$$

The *zero matrix* O, having all elements zero, satisfies

$$A+O = A, \quad \text{and} \quad A+(-A) = O, \tag{5}$$

where $-A = [-a_{ij}]$.

0.1.5 MULTIPLICATION BY AN ELEMENT OF F

If k is an element of F, then

$$kA = [ka_{ij}], \tag{6}$$

with the properties

$$k(A+B) = kA+kB, \tag{7}$$

$$(kl)A = k(lA). \tag{8}$$

0.1.6 MATRIX MULTIPLICATION

If A is a matrix of order $m \times n$ and B a matrix of order $n \times r$, then the matrix product of A and B is the matrix of order $m \times r$ defined by

$$AB = [\sum_{p=1}^{n} a_{ip} b_{pj}]. \tag{9}$$

For the matrices A and B to conform for multiplication in the order AB, the number of columns of A must equal the number of rows of B.

Matrix multiplication is associative, distributive, but not commutative:

$$A(BC) = (AB)C, \tag{10}$$

$$A(B+C) = AB+AC, \quad (A+B)C = AC+BC, \tag{11}$$

$$\text{but} \quad AB \neq BA, \quad \text{in general.} \tag{12}$$

If $AB = BA$, the matrices A and B are said to *commute*; a necessary (but not sufficient) condition for A and B to commute is that they be square matrices of the same order. For matrices

which do commute, the usual formulae valid for elements of F are also valid; e.g.,

$$(A+B)^2 = A^2+2AB+B^2,$$

whereas, in general,

$$(A+B)^2 = A^2+AB+BA+B^2.$$

The *unit matrix* of order n is defined by

$$I_n = I = [\delta_{ij}], \tag{13}$$

where $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$.

It satisfies the relations

$$AI = A, \qquad IB = B, \tag{14}$$

for every matrix A of order $m \times n$ and every matrix B of order $n \times r$. The matrix I commutes with any square matrix of the same order.

0.1.7 Matrix powers

From the associative law of matrix multiplication there follows a well-defined meaning for the pth power of a square matrix A, defined by

$$A^p = AA \ldots A, \quad p \text{ factors}, \tag{15}$$

where p is a positive integer and $A^1 = A$. If, in addition, the zeroth power A^0 is defined to be

$$A^0 = I, \tag{16}$$

then the index law

$$A^p A^q = A^{p+q} \tag{17}$$

holds for all non-negative integers p, q.

A matrix A is called

involutory, if $A^2 = I$; (18)

idempotent, if $A^2 = A$; (19)

nilpotent, if $A^p = O$ for some positive integer p. (20)

0.1.8 PARTITIONED MATRICES

If two matrices A of order $m\times n$ and B of order $n\times r$ are partitioned so that

$$A = \begin{array}{c} \\ m_1 \\ m_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \end{array}, \qquad B = \begin{array}{c} \\ n_1 \\ n_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} r_1 & r_2 \end{array} \\ \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] \end{array}, \tag{21}$$

where

$$m_1+m_2 = m, \qquad n_1+n_2 = n, \qquad r_1+r_2 = r,$$

then

$$AB = \begin{array}{c} \\ m_1 \\ m_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} r_1 & r_2 \end{array} \\ \left[\begin{array}{cc} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{array}\right] \end{array}. \tag{22}$$

The matrices can be partitioned in any way into any number of submatrices, provided that the columns of A are partitioned in the same way as the rows of B; the product AB can then be formed as though the submatrices were elements.

0.1.9 TRANSPOSE

Associated with a matrix $A = [a_{ij}]$ of order $m\times n$ is the matrix A', the *transpose* of A, defined by $A' = [b_{ij}]$, where $b_{ji} = a_{ij}$. The rows (columns) of A' are the columns (rows) of A, and A' is of order $n\times m$. Simple properties of matrix transposition are:

$$(A+B)' = A'+B', \tag{23}$$

$$(kA)' = kA', \tag{24}$$

$$(AB)' = B'A', \tag{25}$$

$$(A^p)' = (A')^p, \tag{26}$$

$$\begin{array}{c} \\ m_1 \\ m_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array}\right]' \end{array} = \begin{array}{c} \\ n_1 \\ n_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} m_1 & m_2 \end{array} \\ \left[\begin{array}{cc} A_{11}' & A_{21}' \\ A_{12}' & A_{22}' \end{array}\right] \end{array}.$$

0.1.10 HERMITIAN CONJUGATE

If the elements a_{ij} of the $m\times n$ matrix A are complex numbers,

the complex conjugate, or *conjugate*, of A is defined to be

$$\bar{A} = [\bar{a}_{ij}]. \tag{27}$$

The *hermitian conjugate* of A is the $n \times m$ matrix $\bar{A}' = [b_{ij}]$, where $\bar{b}_{ji} = a_{ij}$.

§ 0.2 Determinants

0.2.1 DEFINITION. *The determinant of a square matrix* $A = [a_{ij}]$ *of order* n *is defined by*

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & \dots & . \\ . & . & \dots & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

$$= \sum \varepsilon_{ij\dots k}\, a_{i1}\, a_{j2} \dots a_{kn}, \tag{28}$$

where the summation is taken over all values $i, j, \dots, k = 1, 2, \dots, n$ *and* ε *is the alternating symbol defined by*

$$\varepsilon_{ij\dots k} = \begin{cases} +1, \text{ if } (i, j, \dots, k) \text{ is an even permutation of } (1, 2, \dots, n), \\ -1, \text{ if } (i, j, \dots, k) \text{ is an odd permutation of } (1, 2, \dots, n), \\ 0, \text{ otherwise.} \end{cases} \tag{29}$$

0.2.2 ELEMENTARY PROPERTIES

If two columns (rows) of a matrix are interchanged, the sign of its determinant is reversed; hence, if a matrix has two columns (rows) equal, its determinant is zero. A more general formula for the determinant of A is therefore

$$\varepsilon_{pq\dots r}\, |A| = \sum_{i\dots k} \varepsilon_{ij\dots k}\, a_{ip}\, a_{jq} \dots a_{kr}. \tag{30}$$

If each column (row) of a matrix is multiplied by a constant k, the determinant of the matrix is multiplied by k^n; hence

$$|kA| = k^n |A|. \tag{31}$$

If to any column (row) of a matrix, k times another column (row) is added, the determinant of the matrix is unchanged.

If each element in any one column (row) of a matrix is a sum of two terms, the determinant of the matrix can be expressed as the sum of the determinants of two matrices, e.g.,

$$\begin{aligned}&\sum \varepsilon_{ij\ldots k}(b_{i1}+c_{i1})a_{j2}\ldots a_{kn}\\ &=\sum \varepsilon_{ij\ldots k}b_{i1}a_{j2}\ldots a_{kn}+\sum \varepsilon_{ij\ldots k}c_{i1}a_{j2}\ldots a_{kn}.\end{aligned}\tag{32}$$

Transposition of a matrix does not alter its determinant:

$$|A'| = |A|.\tag{33}$$

The determinant of the product of two matrices is the product of their determinants:

$$|AB| = |A|\,|B|.\tag{34}$$

0.2.3 Minors and cofactors

If M_{ij} is the $(n-1)$-square submatrix of A obtained by eliminating the i-th row and the j-th column, then the *minor* of a_{ij} is defined to be $|M_{ij}|$ and the *cofactor* of a_{ij}, denoted by α_{ij}, is defined by

$$\alpha_{ij} = (-1)^{i+j}|M_{ij}|.\tag{35}$$

A determinant can be expanded in terms of the elements of a column (row) and its cofactors:

$$|A| = \sum_i \alpha_{i1}a_{i1},\tag{36}$$

or, more generally,

$$\sum_p \alpha_{pi}a_{pj} = \sum_p a_{ip}\alpha_{jp} = |A|\delta_{ij}.\tag{37}$$

Let $i_1, i_2, \ldots, i_m$ and $j_1, j_2, \ldots, j_m$ signify m rows and m columns of A and $i_{m+1}, \ldots, i_n$ and $j_{m+1}, \ldots, j_n$ signify the remaining rows and columns of A, where $i_1 < i_2 < \ldots < i_m$ and $i_{m+1} < \ldots < i_n$, with a similar ordering of the columns. If $A^{j_1 j_2 \ldots j_m}_{i_1 i_2 \ldots i_m}$ is the m-square submatrix of A obtained by choosing (not eliminating) rows $i_1, i_2, \ldots, i_m$ and columns $j_1, j_2, \ldots, j_m$, then

$$|A^{j_1 j_2 \ldots j_m}_{i_1 i_2 \ldots i_m}| \quad \text{and} \quad |A^{j_{m+1} \ldots j_n}_{i_{m+1} \ldots i_n}|$$

are called *complementary minors*. The *Laplace expansion* of $|A|$ in terms of complementary minors is given by

$$|A| = \sum (-1)^p |A^{j_1 j_2 \cdots j_m}_{i_1 i_2 \cdots i_m}| \, |A^{j_{m+1} \cdots j_n}_{i_{m+1} \cdots i_n}|, \tag{38}$$

where $i_1, i_2, \ldots, i_m$ represents a fixed choice of m rows of A and the summation is taken over all possible choices of m columns, represented by $j_1, j_2, \ldots, j_m$, and $p = i_1 + i_2 + \ldots + i_m + j_1 + j_2 + \ldots + j_m$.

§ 0.3 Inverses

0.3.1 Definitions

If $A = [a_{ij}]$ is a square matrix of order n, the adjoint of A is the matrix of order n defined by

$$\text{adj } A = [b_{ij}], \tag{39}$$

where $b_{ji} = \alpha_{ij} =$ cofactor of a_{ij}. From (37), *it follows that*

$$(\text{adj } A)\, A = A\, (\text{adj } A) = |A| I. \tag{40}$$

A matrix A for which $|A| \neq 0$ is called *non-singular.*

A non-singular matrix has a unique *inverse* A^{-1}, defined by

$$A^{-1} = \frac{1}{|A|} \text{adj } A \tag{41}$$

satisfying

$$A^{-1} A = A A^{-1} = I. \tag{42}$$

0.3.2 Elementary properties

Some basic properties of the inverse of a matrix are

$$(kA)^{-1} = k^{-1} A^{-1}, \tag{43}$$

$$(AB)^{-1} = B^{-1} A^{-1}, \tag{44}$$

$$(A^p)^{-1} = (A^{-1})^p, \tag{45}$$

$$(A')^{-1} = (A^{-1})', \tag{46}$$

$$|A^{-1}| = 1/|A|. \tag{47}$$

§ 0.4 Special Matrices

The following n-square matrices are of special importance in

the theory and applications of matrices.

0.4.1 If $a_{ij} = \begin{cases} 0, & i \neq j \\ k, & i = j \end{cases}$, so that

$$A = \begin{bmatrix} k & 0 & . & . & 0 \\ 0 & k & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & k \end{bmatrix},$$

then A is called a *scalar matrix*, and $A = kI$.

0.4.2 If $a_{ij} = 0$, $i \neq j$, so that

$$A = \begin{bmatrix} a_{11} & 0 & & 0 \\ 0 & a_{22} & & 0 \\ . & . & & . \\ . & . & & . \\ 0 & 0 & & a_{nn} \end{bmatrix},$$

then A is called a *diagonal matrix*.

0.4.3 If $a_{ij} = 0$, $i > j$, so that

$$A = \begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ 0 & a_{22} & & a_{2n} \\ . & . & & . \\ . & . & & . \\ 0 & 0 & & a_{nn} \end{bmatrix},$$

then A is called an *upper triangular matrix*, usually denoted by U. Similarly, if $a_{ij} = 0$, $i < j$, then A is called a *lower triangular matrix*, usually denoted by L.

0.4.4 In terms of the transpose of A, the following special matrices are defined:

$$\textit{symmetric } A' = A, \tag{48}$$

$$\textit{skew-symmetric } A' = -A, \tag{49}$$

$$\textit{orthogonal } A' = A^{-1}. \tag{50}$$

0.4.5 In terms of the hermitian conjugate of A, the following special matrices are defined:

$$\textit{hermitian } \bar{A}' = A, \tag{51}$$

$$\textit{skew-hermitian } \bar{A}' = -A, \tag{52}$$

$$\textit{unitary } \bar{A}' = A^{-1}. \tag{53}$$

Note that:

a real symmetric matrix is hermitian,
a real skew-symmetric matrix is skew-hermitian,
a real orthogonal matrix is unitary.

0.4.6 If A is a hermitian matrix with its leading principal minors

$$p_1 = |A_1^1| = a_{11}, \quad p_2 = |A_{12}^{12}| = \left|\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}\right|, \ldots, p_n = |A|$$

all positive, then A is called *positive definite.*

Associated with a hermitian matrix A is the real *hermitian form* $\bar{x}'Ax = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\bar{x}_i x_j$, where the elements of the column x are complex numbers x_i. A necessary and sufficient condition for A to be positive definite is that $\bar{x}'Ax > 0$ for all $x \neq O$.

Associated with a real symmetric matrix A is the *quadratic form* $x'Ax = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}x_i x_j$, where the elements of the column x are real numbers x_i. A necessary and sufficient condition for A to be positive definite is that $x'Ax > 0$ for all $x \neq O$.

0.4.7 A matrix which commutes with its hermitian conjugate, that is

$$A\bar{A}' = \bar{A}'A, \tag{54}$$

is called a *normal matrix.*

A triangular matrix (upper or lower) which is normal is diagonal.

§ 0.5 Rank, Linear Forms and Equations

0.5.1 Rank

Consider the linear relation

$$\begin{aligned} & k_1[a_{11} \quad a_{12} \quad \quad a_{1n}] \\ +& k_2[a_{21} \quad a_{22} \quad \quad a_{2n}] \\ +& \ldots \\ +& k_m[a_{m1} \quad a_{m2} \quad \ldots \quad a_{mn}] \\ =& \quad [0 \quad 0 \quad \ldots \quad 0]. \end{aligned} \tag{55}$$

If such a relation is valid for certain values of $k_1, k_2, \ldots, k_m$, not all zero, then the m rows are called *linearly dependent.*

If such a relation implies that $k_1 = k_2 = \ldots = k_m = 0$, then the rows are called *linearly independent.*

Similar definitions apply to columns.

Given a matrix A of order $m \times n$,

the *row rank* of A is the maximum number of linearly independent rows of A;

the *column rank* of A is the maximum number of linearly independent columns of A.

The *determinant* rank of A is defined to be the number of rows in the largest square submatrix of A which has non-zero determinant.

An important result is the following:

row rank = column rank = determinant rank.

The rank of the product of two matrices is not greater than the rank of either factor.

The rank of a matrix is unaltered, if it is multiplied by a non-singular matrix.

A square matrix is non-singular if and only if its rank is equal to the order of the matrix.

0.5.2 Linear forms

Consider the m linear forms in n variables

$$\begin{aligned} f_1 &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n, \\ f_2 &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n, \\ &\vdots \\ f_m &= a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n, \end{aligned} \tag{56}$$

or, in matrix notation,

$$f = Ax. \tag{57}$$

The forms are called *linearly dependent*, if for all $x_1, \ldots, x_n$,

$$k_1 f_1 + k_2 f_2 + \ldots + k_m f_m = 0 \tag{58}$$

for some non-trivial $k_1, k_2, \ldots, k_m$ (i.e. not all zero); otherwise the forms are *linearly independent*. The linear relation (58) is equivalent to the linear relation (55) between the rows of the coefficient matrix A.

If the linear forms (or the rows) are linearly dependent, the linear relation (58) is determined from

$$k'A = O \quad \text{or} \quad A'k = O, \tag{59}$$

where k is the column with elements $k_1, k_2, \ldots, k_m$.

0.5.3 Linear equations

The linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2, \\ &\cdot \\ &\cdot \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m, \end{aligned} \tag{60}$$

can be written in matrix notation as

$$Ax = b. \tag{61}$$

If $b = O$, the equations are called *homogeneous*, otherwise *non-homogeneous*.

The *augmented matrix* $[Ab]$ is the matrix of order $m \times (n+1)$, defined by

$$[Ab] = \begin{bmatrix} a_{11} & a_{12} & & a_{1n} & b_1 \\ a_{21} & a_{22} & & a_{2n} & b_2 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ a_{m1} & a_{m2} & & a_{mn} & b_m \end{bmatrix}. \tag{62}$$

The equations $Ax = b$ have a solution if and only if A and $[Ab]$ have the same rank.

If A is a non-singular n-square matrix, the unique solution of the equations $Ax = b$ is $x = A^{-1}b$.

If the rank of the matrix A of order $m \times n$ is equal to r, the number of linearly independent solutions of the homogeneous equations $Ax = O$ is $n-r$.

§ 0.6 Eigenvalues and Eigenvectors

0.6.1 Definitions

Let A be an n-square matrix. If

$$Ax = \lambda x, \qquad x \neq O, \tag{63}$$

then x is called an eigenvector of A and λ the corresponding eigenvalue of A. The eigenvector x can be multiplied by a non-zero constant, since (63) implies that $A(kx) = \lambda(kx)$. The eigenvector can be *normalised* so that $x'x = 1$ (if x is real), or $\bar{x}'x = 1$ (if x is complex).

The eigenvalues are the zeros of the characteristic polynomial

$$\phi(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \cdot & \cdot & & \\ \cdot & \cdot & & \cdot \\ -a_{n1} & -a_{n2} & & \lambda - a_{nn} \end{vmatrix}$$

$$= \lambda^n + p_1\lambda^{n-1} + \ldots + p_{n-1}\lambda + p_n. \tag{64}$$

The matrix A has at most n distinct eigenvalues. If λ_1 is a simple root of $\phi(\lambda) = 0$, λ_1 is called a *non-degenerate* eigenvalue; if the multiplicity of the root is r, the *degeneracy* of λ_1 is r.

0.6.2 Elementary properties

The sum of the eigenvalues $= a_{11} + a_{22} + \ldots + a_{nn} =$ trace of $A =$ tr A.

The product of the eigenvalues $= |A|$.

The eigenvalues of kA are $k\lambda$ and the eigenvectors of kA are the same as those of A,

$$\text{since } Ax = \lambda x \text{ implies } kAx = k\lambda x. \tag{65}$$

The eigenvalues of A^p are λ^p and the eigenvectors are the same as those of A,

$$\text{since } Ax = \lambda x \text{ implies } A^p x = \lambda^p x. \tag{66}$$

If

$$P(A) = c_0 I + c_1 A + c_2 A^2 + \ldots + c_m A^m \tag{67}$$

is a polynomial in A, then

$$P(A)x = P(\lambda)x, \tag{68}$$

so that the eigenvalues of $P(A)$ are $P(\lambda)$ and the eigenvectors of A are also eigenvectors of $P(A)$.

If A is non-singular, the eigenvalues of A^{-1} are λ^{-1} and the eigenvectors of A^{-1} are the same as those of A, since $Ax = \lambda x$ implies $A^{-1}x = \lambda^{-1}x$.

The eigenvalues of A' are the same as those of A, but the eigenvectors are, in general, different.

If P is a non-singular matrix the eigenvalues of $P^{-1}AP$ are the same as those of A and the eigenvectors are $P^{-1}x$, since

$$Ax = \lambda x \text{ implies } P^{-1}AP\ P^{-1}x = \lambda P^{-1}x. \tag{69}$$

[The matrix $P^{-1}AP$ is said to be *similar* to A].

Eigenvectors corresponding to different eigenvalues are linearly independent.

If A has n linearly independent eigenvectors, the matrix P with these as columns is non-singular, and $P^{-1}AP$ is a diagonal matrix with the eigenvalues of A as diagonal elements.

0.6.3 Cayley-Hamilton theorem

A matrix satisfies its own characteristic equation; more precisely, if

$$\phi(\lambda) = |\lambda I - A| = \lambda^n + p_1 \lambda^{n-1} + \ldots + p_{n-1}\lambda + p_n,$$

then $\phi(A) = A^n + p_1 A^{n-1} + \ldots + p_{n-1} A + p_n I = O$.

Polynomials in A of any degree can be expressed as polynomials in A of degree $\leqq n-1$. If A is non-singular, all negative powers of A can be likewise expressed as polynomials in A of degree $\leqq n-1$.

0.6.4 Special matrices

The eigenvalues and eigenvectors of the special n-square

matrices defined in **0.4** have the following properties:

(1) The eigenvalues of the *scalar matrix* $A = kI$ are $\lambda = k$ (n-fold degenerate) and any column x is an eigenvector.

(2) The eigenvalues of a *diagonal matrix* are the diagonal elements, i.e., $\lambda_i = a_{ii}$, and the unit columns

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \ldots, \qquad e_n = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 1 \end{bmatrix},$$

are eigenvectors.

(3) The eigenvalues of a *triangular matrix* (upper or lower) are the elements on the leading diagonal, i.e., $\lambda_i = a_{ii}$. The column e_1 is an eigenvector corresponding to $\lambda_1 = a_{11}$ for the upper triangular matrix. The column e_n is an eigenvector corresponding to $\lambda_n = a_{nn}$ for the lower triangular matrix. Other eigenvectors are more complicated.

(4) The eigenvalues of a *real symmetric matrix* are real and the eigenvectors are real orthogonal in the sense that if x and y are eigenvectors of A corresponding to different eigenvalues, then $x'y = 0$.

(5) The eigenvalues of a *positive definite real symmetric matrix* are positive.

(6) The eigenvalues of a *real skew-symmetric matrix* are either 0 or occur in pairs $\pm i\theta_j$, θ_j real. The eigenvectors are unitary in the sense that if x and y are eigenvectors of A corresponding to different eigenvalues, then $\bar{x}'y = 0$.

(7) The eigenvalues of a *real orthogonal matrix* are either **1**, -1, or occur in pairs $\exp(\pm i\theta_j)$. The eigenvectors are unitary.

(8) The eigenvalues of a *hermitian matrix* are real and the eigenvectors unitary.

(9) The eigenvalues of a *positive definite hermitian matrix* are positive.

(10) The eigenvalues of a *skew-hermitian matrix* are pure imaginary, i.e., of the form $i\theta_j$, and the eigenvectors unitary.

(11) The eigenvalues of a *unitary matrix* are of the form $\exp(i\theta_j)$ and the eigenvectors are unitary.
(12) The eigenvectors of a *normal matrix* are unitary.

§ 0.7 Elementary Canonical Forms

0.7.1 Equivalent transformation

If A is an $m \times n$ matrix, Q is an m-square non-singular matrix, P is an n-square non-singular matrix, then QAP is called an *equivalent transformation* of the matrix A and QAP is called *equivalent* to A, denoted by

$$QAP \sim A. \tag{70}$$

Equivalent matrices have the same rank.

If the rank of matrix A is r, there exists an equivalent transformation which reduces A to the *canonical form*

$$QAP = \begin{array}{c} \\ r \\ m-r \end{array} \begin{array}{c} \begin{array}{cc} r & n-r \end{array} \\ \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} \end{array}. \tag{71}$$

The matrix A can be reduced to this canonical form by a sequence of *elementary operations* on its rows and columns, viz.,

interchange of two rows (columns),

multiplying a row (column) by a non-zero scalar,

adding to a row (column) a scalar multiple of another row (column).

Each elementary operation can be represented by an *elementary matrix* which may be obtained by performing the elementary operation on the unit matrix.

If A is a non-singular n-square matrix, there exists an equivalent transformation which reduces A to the canonical form

$$QAP = I_n, \tag{72}$$

where Q and P are products of elementary matrices.

0.7.2 Congruent transformation

If A is an n-square matrix and P is a non-singular n-square

matrix, then $P'AP$ is called a *congruent transformation* of the matrix A and $P'AP$ is called *congruent* to A, denoted by

$$P'AP \overset{c}{\sim} A. \tag{73}$$

The congruent transformation is a special case of an equivalent transformation.

Matrices congruent to a symmetric matrix are symmetric; and matrices congruent to a skew-symmetric matrix are skew-symmetric.

If A is a complex symmetric matrix of rank r, there exists a congruent transformation with a complex matrix P which reduces A to the canonical form

$$P'AP = \begin{array}{c} \\ r \\ n-r \end{array} \begin{array}{c} \begin{array}{cc} r & n-r \end{array} \\ \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} \end{array}. \tag{74}$$

If A is a real symmetric matrix of rank r, there exists a congruent transformation with a real matrix P which reduces A to the canonical form

$$P'AP = \begin{array}{c} \\ p \\ q \\ n-r \end{array} \begin{array}{c} \begin{array}{ccc} p & q & n-r \end{array} \\ \begin{bmatrix} I_p & O & O \\ O & -I_q & O \\ O & O & O \end{bmatrix} \end{array}, \tag{75}$$

where $p+q = r$. The numbers p, q define the *signature* (p, q) of the real symmetric matrix A, and the signature is invariant under congruent transformations.

A real symmetric matrix A can be reduced to its canonical form by reducing the corresponding quadratic form $x'Ax$ to the form $y_1^2 + \ldots + y_p^2 - y_{p+1}^2 - \ldots - y_{p+q}^2$ by an appropriate change in variable $x = Py$.

The signature of a positive definite real symmetric matrix A of order n is $(n, 0)$, and there exists for such a matrix a congruent transformation with a real non-singular upper triangular matrix U which reduces A to the canonical form

$$U'AU = I_n. \tag{76}$$

If A is an n-square positive definite real symmetric matrix and B an n-square real symmetric matrix, there exists a congruent transformation with a real non singular matrix P which reduces A and B simultaneously to the canonical forms

$$P'AP = I_n, \qquad P'BP = \begin{bmatrix} \lambda_1 & 0 & . & 0 \\ 0 & \lambda_2 & . & 0 \\ . & . & . & . \\ . & . & . & . \\ 0 & 0 & . & \lambda_n \end{bmatrix}, \tag{77}$$

where the λ_i are the eigenvalues of $A^{-1}B$.

0.7.3 Hermitian congruent transformation

If A is an n-square matrix and P is a non-singular n-square matrix, then $\bar{P}'AP$ is called a *hermitian congruent transformation* of the matrix A and $\bar{P}'AP$ is called *hermitian congruent* to A, denoted by

$$\bar{P}'AP \overset{hc}{\sim} A. \tag{78}$$

The hermitian congruent transformation is a special case of an equivalent transformation.

Matrices hermitian congruent to a hermitian matrix are hermitian; and matrices hermitian congruent to a skew-hermitian matrix are skew-hermitian.

If A is a hermitian matrix of rank r, there exists a hermitian congruent transformation which reduces A to the canonical form

$$\bar{P}'AP = \begin{bmatrix} I_p & O & O \\ O & -I_q & O \\ O & O & O \end{bmatrix}, \tag{79}$$

where $p+q = r$ and (p, q) is the *signature* of A. The signature is invariant under hermitian congruent transformations.

The signature of a positive definite hermitian matrix A of order n is $(n, 0)$ and there exists for such a matrix a hermitian congruent transformation with a non-singular upper triangular matrix U which reduces A to the canonical form

$$\bar{U}'AU = I_n. \tag{80}$$

If A is an n-square positive definite hermitian matrix and B an n-square hermitian matrix, there exists a hermitian congruent transformation with a non-singular matrix P which reduces A and B simultaneously to the canonical forms

$$\bar{P}'AP = I_n, \qquad \bar{P}'BP = \begin{bmatrix} \lambda_1 & 0 & . & 0 \\ 0 & \lambda_2 & . & 0 \\ . & . & . & . \\ . & . & . & . \\ 0 & 0 & . & \lambda_n \end{bmatrix}, \tag{81}$$

where the λ_i are the eigenvalues of $A^{-1}B$.

0.7.4 Unitary transformation

If A is an n-square matrix and P is an n-square unitary matrix, then $\bar{P}'AP$ is called a *unitary transformation* of A. A unitary transformation is a special case of a hermitian congruent transformation.

For any matrix A there exists a unitary transformation which reduces A to the canonical form

$$\bar{P}'AP = U = \begin{bmatrix} \lambda_1 & * & . & . & * \\ 0 & \lambda_2 & . & . & * \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & \lambda_n \end{bmatrix}, \tag{82}$$

where U is an upper triangular matrix the diagonal elements of which are the eigenvalues λ_i of A. For a normal matrix A there exists a unitary transformation which reduces A to the canonical form

$$\bar{P}'AP = D = \begin{bmatrix} \lambda_2 & 0 & . & 0 \\ 0 & \lambda_2 & . & 0 \\ . & . & . & . \\ . & . & . & . \\ 0 & 0 & . & \lambda_n \end{bmatrix}, \tag{83}$$

where D is a diagonal matrix the diagonal elements of which are the eigenvalues λ_i of A. The columns of P are normalised eigenvectors of A.

A necessary and sufficient condition for it to be possible to

diagonalise a matrix A by a unitary transformation is that A be normal.

0.7.5 REAL ORTHOGONAL TRANSFORMATION

If A is an n-square matrix and P a real orthogonal n-square matrix, then $P'AP$ is called a *real orthogonal transformation* of A. A real orthogonal transformation is a special case of a unitary transformation.

If A is a real symmetric matrix, there exists a real orthogonal transformation which reduces A to the canonical form

$$P'AP = D = \begin{bmatrix} \lambda_1 & 0 & . & . & 0 \\ 0 & \lambda_2 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & \lambda_n \end{bmatrix}, \tag{84}$$

where D is a diagonal matrix the diagonal elements of which are the eigenvalues of A. The columns of P are the (real) normalised eigenvectors of A.

If A is a real orthogonal matrix with eigenvalues 1 (p-fold degenerate), -1 (q-fold degenerate) and $\exp(\pm i\theta_1), \ldots,$ $\exp(\pm i\theta_r)$, then there exists a real orthogonal transformation which reduces A to the canonical form

$$P'AP = \begin{bmatrix} I_p & O & O & O \\ O & -I_q & O & O \\ O & O & D_1 & O \\ . & . & . & . \\ O & O & O & D_r \end{bmatrix},$$

where

$$D_j = \begin{bmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{bmatrix}, \quad j = 1, 2, \ldots, r. \tag{85}$$

0.7.6 SIMILARITY TRANSFORMATION

If A is an n-square matrix and P a non-singular n-square matrix, then $P^{-1}AP$ is called a *similarity transformation* of A. Unitary and real orthogonal transformations are special cases of similarity transformations.

If the eigenvalues λ_i of A are non-degenerate, there exists a similarity transformation which reduces A to the canonical form

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & . & 0 \\ 0 & \lambda_2 & . & 0 \\ . & . & . & . \\ . & . & . & . \\ 0 & 0 & . & \lambda_n \end{bmatrix}. \tag{86}$$

The columns of P are eigenvectors of A and the rows of P^{-1} eigenvectors of A'. If A has degenerate eigenvalues, it can be diagonalised if and only if to an r-fold degenerate eigenvalue there correspond r linearly independent eigenvectors.

0.7.7 Factorisation of a matrix

Several of the theorems on canonical forms of a non-singular matrix can be reinterpreted in terms of the factorisation of the matrix as a matrix product. For example, it follows from (72) that, if A is non singular, there exist non-singular matrices $R(= Q^{-1})$ and $S(= P^{-1})$ which are products of elementary matrices and such that

$$A = RS. \tag{87}$$

From (76) it follows that, if A is a positive definite real symmetric matrix, there exists a real non-singular upper triangular matrix $V(= U^{-1})$ such that

$$A = V'V. \tag{88}$$

And from (80) it follows that, if A is a positive definite hermitian matrix, there exists a non-singular upper triangular matrix $V(= U^{-1})$ such that

$$A = \bar{V}'V. \tag{89}$$

More generally, if A is a non-singular matrix with its rows arranged so that its leading principal minors $p_1, p_2, \ldots, p_n$ are non-zero (this arrangement being always possible for a non-singular matrix), then there exist

a lower triangular matrix L, diagonal elements 1,

a diagonal matrix D, diagonal elements $p_1, \dfrac{p_2}{p_1}, \ldots, \dfrac{p_n}{p_{n-1}}$,

an upper triangular matrix U, diagonal elements 1, such that

$$A = LDU. \tag{90}$$

The matrix D may be absorbed into L giving the factorisation

$$A = LU, \tag{91}$$

where L is a lower triangular matrix, and U is an upper triangular matrix, diagonal elements 1.

If A is a positive definite real symmetric matrix, then $p_i > 0$ and $D = D_1^2$, where D_1 is the diagonal matrix with elements $\sqrt{p_1}, \sqrt{(p_2/p_1)}, \ldots, \sqrt{(p_n/p_{n-1})}$. In (90), L becomes U', since A is symmetric, and by defining the real non-singular upper triangular matrix $V = D_1 U$, the factorisation becomes

$$A = V'V,$$

as given by (88).

§ 1. Products and Powers

1.1 Prove that the following set of matrices (the Pauli spin matrices used in quantum theory)

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

satisfy the relations

(a) $A^2 = B^2 = C^2 = I$,

(b) $BC = -CB = iA$, $CA = -AC = iB$, $AB = -BA = iC$.

1.2 If

$$A(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix},$$

prove that

$$A(\alpha)A(\beta) = A(\alpha+\beta).$$

1.3 Find the matrix A of order 2 which satisfies the relation

$$\begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} A \begin{bmatrix} -1 & 2 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}.$$

1.4 Find all matrices which commute with

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, (b) $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

HINT. (a) Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a matrix such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Deduce that

$$2c = 3b, \qquad d = a+c,$$

and hence that the matrices are of the form

$$p\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + q\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

1.5 Find all matrices A^k, k a positive integer, if

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

HINT. Note that $A^4 = I$ so that A^k is equal to one of A, A^2, A^3, I.

1.6 If k is a positive integer, evaluate

$$\text{(a)} \quad \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}^k, \qquad \text{(b)} \quad \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix}^k.$$

HINT. (a) Write

$$\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} = aI + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and use the binomial theorem.

1.7 If k is a positive integer, prove that

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^k = \begin{bmatrix} a^k & b(a^{k-1}+a^{k-2}c+\ldots+ac^{k-2}+c^{k-1}) \\ 0 & c^k \end{bmatrix}.$$

1.8 If k is a positive integer, evaluate A^k if

$$\text{(a)} \ A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}, \qquad \text{(b)} \ A = \begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix}.$$

HINT. (a) In one method of solution, A^k is written as

$$A^k = \begin{bmatrix} a(k) & b(k) \\ c(k) & d(k) \end{bmatrix}, \qquad k = 0, 1, \ldots$$

where the unknown functions $a(k), \ldots, d(k)$ are easily shown, from the identity $A^{k+1} = A^kA$, to satisfy the first order simultaneous difference equations

$$a(k+1) = 4a(k)+b(k)$$
$$b(k+1) = 2a(k)+3b(k)$$
$$c(k+1) = 4c(k)+d(k)$$
$$d(k+1) = 2c(k)+3d(k).$$

Elimination of $b(k)$ from the first two equations gives the second order difference equation

$$a(k+2)-7a(k+1)+10a\ (k) = 0,$$

the general solution of which is

$$a(k) = \alpha_1 5^k+\alpha_2 2^k,$$

giving $b(k) = \alpha_1 5^k-2\alpha_2 2^k$. Since $A^0 = I$, $a(0) = 1$ and $b(0) = 0$, so that $\alpha_1 = \frac{2}{3}$, $\alpha_2 = \frac{1}{3}$. Similarly,

$$c(k) = \alpha_3 5^k+\alpha_4 2^k$$
$$d(k) = \alpha_3 5^k-2\alpha_4 2^k,$$

and $\alpha_3 = \frac{1}{3}$, $\alpha_4 = -\frac{1}{3}$.

1.9 If k is a positive integer, prove that

(a) $$\begin{bmatrix} \cos\alpha+\sin\alpha & 2\sin\alpha \\ -\sin\alpha & \cos\alpha-\sin\alpha \end{bmatrix}^k = \begin{bmatrix} \cos k\alpha+\sin k\alpha & 2\sin k\alpha \\ -\sin k\alpha & \cos k\alpha-\sin k\alpha \end{bmatrix},$$

(b) $$\begin{bmatrix} -1 & 2 \\ 4 & 1 \end{bmatrix}^{2k} = \begin{bmatrix} 3^{2k} & 0 \\ 0 & 3^{2k} \end{bmatrix}.$$

1.10 Given the matrices

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

prove that

$$\sum \varepsilon_{pqrs} E_p E_q E_r E_s = 0,$$

where ε_{pqrs} is the alternating symbol.

1.11 Find all matrices of order 2 for which

(a) $A^2 = I$, (b) $A^2 = O$.

HINT. (a) Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

so that $A^2 = I$ implies

$$a^2+bc = 1, \quad ab+bd = 0, \quad ac+cd = 0, \quad bc+d^2 = 1.$$

From the first and last of these relations, $d^2 = a^2$.

If $d = a \neq 0$, the other relations give $b = c = 0$, so that $a = d = \pm 1$ and $A = \pm I$.

If $d = -a$, then $a^2+bc = 1$, so that

$$A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \text{ with } a^2+bc = 1.$$

Included in this form is the special case

$$A = \begin{bmatrix} 0 & b \\ b^{-1} & 0 \end{bmatrix}, \quad b \neq 0.$$

1.12 Find all matrices which commute with the matrix

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

1.13 If k is a positive integer, evaluate

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^k.$$

1.14 If $P(A)$ is a polynomial in the matrix

$$A = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix},$$

prove that

$$P(A) = \begin{bmatrix} P(a) & P'(a) & \frac{1}{2}P''(a) \\ 0 & P(a) & P'(a) \\ 0 & 0 & P(a) \end{bmatrix}.$$

1.15 If k is a positive integer and

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

prove that

$$A^{k+2} = A^k + A^2 - I.$$

1.16 If

$$A = \tfrac{1}{9}\begin{bmatrix} 4 & 7 & -4 \\ 1 & 4 & 8 \\ 8 & -4 & 1 \end{bmatrix},$$

prove that $A^4 = I$.

1.17 If k is a positive integer and

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix},$$

evaluate A^k.

HINT. Write $A = I + B$ and use the binomial theorem, noting that $B^r = 0$ when $r > 2$.

1.18 If each element of a matrix A of order n is equal to unity, prove that

$$(A/n)^k = A/n,$$

where k is a positive integer.

1.19 If $H = [h_{ij}]$ is a matrix of order n and

$$h_{ij} = \begin{cases} 1, & i = j-1, \\ 0, & i \neq j-1, \end{cases}$$

(a) determine the matrices H^r, r a positive integer;
(b) prove that

$$(H'H)^2 = H'H, \qquad (HH')^2 = HH',$$
$$H^r H'^r + H'^s H^s = I, \qquad r+s = n.$$

HINT. Evaluate the matrices $H^2, H^3, \ldots, H^{n-1}$ and show that $H^n = O$.

1.20 If $A = [a_{ij}]$ is a matrix of order n and

$$a_{ij} = \begin{cases} a, & i = j, \\ 1, & i = j-1, \\ 0, & i \neq j, \quad i \neq j-1, \end{cases}$$

prove that for any positive integer k

$$A^k = \begin{bmatrix} a^k & \binom{k}{1}a^{k-1} & \binom{k}{2}a^{k-2} & \cdot & \cdot & \binom{k}{n-1}a^{k-n+1} \\ 0 & a^k & \binom{k}{1}a^{k-1} & \cdot & \cdot & \binom{k}{n-2}a^{k-n+2} \\ 0 & 0 & a^k & \cdot & \cdot & \binom{k}{n-3}a^{k-n+3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & a^k \end{bmatrix}.$$

HINT. Use the fact that $A = aI+H$, where H is the matrix defined in the previous problem.

1.21 The elements a_{ij} of a matrix A of order n are given by

$$a_{ij} = \begin{cases} 1, & i+j = n+1, \\ 0, & i+j \neq n+1. \end{cases}$$

Prove that $A' = A$ and $A^2 = I$.

1.22 The elements a_{ij} of a matrix A of order n are given by

$$a_{ij} = \begin{cases} (-1)^{j-1}\binom{j-1}{i-1}, & i \leqq j, \\ 0, & i > j. \end{cases}$$

Prove that $A^2 = I$.

1.23 The elements a_{ij} of a matrix A of order $2n$ are given by

$$a_{ij} = \begin{cases} 1, & i = j-1, \\ -1, & i = 2n, \quad j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Prove that $A^{2n} = -I$.

1.24 If Z_1, Z_2 are matrices of order n such that

$$Z_1+Z_2 = I,$$

prove that Z_1, Z_2 are idempotent if and only if $Z_1Z_2 = O$.

HINT. If $Z_1 Z_2 = O$, it follows from

$$Z_1(Z_1+Z_2) = Z_1$$

that $Z_1^2 = Z_1$. Conversely, if $Z_1^2 = Z_1$ and $Z_2^2 = Z_2$, then

$$(Z_1+Z_2)^2 = Z_1+Z_2+Z_1Z_2+Z_2Z_1 = I,$$

giving

$$Z_1Z_2+Z_2Z_1 = O.$$

From premultiplication and postmultiplication of this equation by Z_1 it follows that $Z_1Z_2 = O$.

1.25 If Z_1, Z_2, Z_3 are matrices of order n such that

$$Z_1+Z_2+Z_3 = I,$$

prove that Z_1, Z_2, Z_3 are idempotent if and only if $Z_1Z_2 = Z_2Z_3 = Z_3Z_1 = O$.

HINT. If the matrices are idempotent, prove that Z_1+Z_2 is idempotent and hence, as in the previous problem, that $Z_1Z_2 = O$.

1.26 If A, B are matrices of order n and A commutes with $AB-BA$, prove that if r, s are positive integers, $r > s$, then

(a) $$A^rB-BA^r = r(AB-BA)A^{r-1};$$

(b) $$s(A^rB-BA^r) = r(A^sB-BA^s)A^{r-s}.$$

HINT. (a) Use induction, noting that

$$A^rB-BA^r = A(A^{r-1}B-BA^{r-1})+(AB-BA)A^{r-1}.$$

1.27 The commutator $[A, B]$ of two matrices A, B of order n is defined to be the matrix

$$[A, B] = AB-BA.$$

Prove that

(a) $[A, B] = -[B, A]$;

(b) $[A, [B, C]]+[B, [C, A]]+[C, [A, B]] = O$;

(c) $$[A, [A, [\ldots [A, \underbrace{B]]] \ldots]}_{n} = \sum_{r=0}^{n} (-1)^r \binom{n}{r} A^{n-r}BA^r.$$

HINT. (c) Check for $n = 2, 3$ to help clarify the notation. For a formal proof by induction, the left-hand side is written as

$$A\left\{\sum_0^{n-1}(-1)^r\binom{n-1}{r}A^{n-1-r}BA^r\right\}-\left\{\sum_0^{n-1}(-1)^r\binom{n-1}{r}A^{n-1-r}BA^r\right\}A$$

$$=\sum_0^{n-1}(-1)^r\binom{n-1}{r}A^{n-r}BA^r+\sum_1^{n}(-1)^r\binom{n-1}{r-1}A^{n-r}BA^r,$$

which gives the right-hand side on use of the identity

$$\binom{n-1}{r}+\binom{n-1}{r-1}=\binom{n}{r}.$$

§ 2. Determinants

Establish the following results:

2.1 $$\begin{vmatrix} 1 & 2 & 0 & 1 \\ 2 & 3 & -1 & 0 \\ 0 & -1 & 2 & 4 \\ -1 & 0 & 4 & -1 \end{vmatrix} = -2.$$

2.2 $$\begin{vmatrix} 1 & -1 & 3 & -4 \\ -1 & 2 & 0 & 4 \\ 3 & 0 & 4 & -2 \\ -4 & 5 & -2 & 0 \end{vmatrix} = 154.$$

2.3 $$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 \end{vmatrix} = -160.$$

2.4 $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b).$$

2.5 $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (b-a)(c-a)(c-b)(a+b+c).$$

2.6 $$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (b-a)(c-a)(c-b)(ab+bc+ca).$$

2.7 $$\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 \\ b^2 & (b+1)^2 & (b+2)^2 \\ c^2 & (c+1)^2 & (c+2)^2 \end{vmatrix} = 4(a-b)(a-c)(b-c).$$

2.8 $$\begin{vmatrix} 1 & bc+ad & b^2c^2+a^2d^2 \\ 1 & ca+bd & c^2a^2+b^2d^2 \\ 1 & ab+cd & a^2b^2+c^2d^2 \end{vmatrix} = (b-a)(c-a)(d-a)(c-b) \times (d-b)(d-c).$$

2.9 $$\begin{vmatrix} s_k & s_{k+1} & s_{k+2} \\ s_{k+1} & s_{k+2} & s_{k+3} \\ s_{k+2} & s_{k+3} & s_{k+4} \end{vmatrix} = a^k b^k c^k (b-a)^2 (c-a)^2 (c-b)^2,$$

where $s_k = a^k+b^k+c^k$.

2.10 $$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = (b-a)(c-a)(d-a)(c-b)(d-b) \times (d-c).$$

2.11 $$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^4 & b^4 & c^4 & d^4 \end{vmatrix} = (b-a)(c-a)(d-a)(c-b)(d-b) \times (d-c)(a+b+c+d).$$

2.12 $$\begin{vmatrix} 1 & a+b+c & ab+bc+ca & abc \\ 1 & b+c+d & bc+cd+db & bcd \\ 1 & c+d+a & cd+da+ac & cda \\ 1 & d+a+b & da+ab+bd & dab \end{vmatrix} = -(b-a)(c-a)(d-a) \times (c-b)(d-b)(d-c).$$

2.13 $$\begin{vmatrix} abcd & abd & acd & bcd \\ abc & abcd & acd & bcd \\ abc & abd & abcd & bcd \\ abc & abd & acd & abcd \end{vmatrix} = a^3 b^3 c^3 d^3 (abcd-ab-bc-cd -da+2a+2b+2c+2d-3).$$

2.14 $$\begin{vmatrix} -2a & a+b & c+a \\ a+b & -2b & b+c \\ c+a & b+c & -2c \end{vmatrix} = 4(a+b)(b+c)(c+a).$$

2.15 $$\begin{vmatrix} (a+b)^2 & b^2 & a^2 \\ b^2 & (b+c)^2 & c^2 \\ a^2 & c^2 & (c+a)^2 \end{vmatrix} = 2(ab+bc+ca)^3.$$

2.16 $$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3.$$

2.17 $$\begin{vmatrix} 2a+b+c & b & c \\ a & a+2b+c & c \\ a & b & a+b+2c \end{vmatrix} = 2(a+b+c)^3.$$

2.18 $$\begin{vmatrix} a & b & c & d \\ a & -b & -c & -d \\ a & b & -c & -d \\ a & b & c & -d \end{vmatrix} = -8abcd.$$

2.19 $$\begin{vmatrix} -a & b & c & d \\ b & -a & d & c \\ c & d & -a & b \\ d & c & b & -a \end{vmatrix} = -(b+c+d-a)(c+d+a-b) \times (d+a+b-c)(a+b+c-d).$$

2.20 $$\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix} = (a^2+b^2+c^2+d^2)^2.$$

2.21 $$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix} = (af-be+cd)^2.$$

2.22 $$\begin{vmatrix} 1 & x & x^2 \\ \cos a\theta & \cos(a+1)\theta & \cos(a+2)\theta \\ \sin a\theta & \sin(a+1)\theta & \sin(a+2)\theta \end{vmatrix} = (1-2x\cos\theta+x^2)\sin\theta.$$

2.23 $$\begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \cos 2\alpha & \cos 2\beta & \cos 2\gamma \\ \cos 3\alpha & \cos 3\beta & \cos 3\gamma \end{vmatrix} = 4(\cos\beta-\cos\alpha)(\cos\gamma-\cos\alpha) \times (\cos\gamma-\cos\beta)(\cos\alpha+\cos\beta +\cos\gamma+2\cos\alpha\cos\beta\cos\gamma).$$

2.24 $$\begin{vmatrix} 1 & \cos\alpha & \cos\alpha \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix} = 4\sin\tfrac{1}{2}(\alpha+\beta+\gamma)\sin\tfrac{1}{2}(\alpha+\beta-\gamma) \times \sin\tfrac{1}{2}(\beta+\gamma-\alpha)\sin\tfrac{1}{2}(\gamma+\alpha-\beta).$$

2.25
$$\begin{vmatrix} 1 & \sin(\alpha+\beta) & \cos(\alpha-\beta) \\ 1 & \sin(\beta+\gamma) & \cos(\beta-\gamma) \\ 1 & \sin(\gamma+\alpha) & \cos(\gamma-\alpha) \end{vmatrix} = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos 2\alpha \\ \sin\beta & \cos\beta & \cos 2\beta \\ \sin\gamma & \cos\gamma & \cos 2\gamma \end{vmatrix}.$$

2.26
$$\begin{vmatrix} 1 & \sin 2\alpha & \cos 2\alpha \\ 1 & \sin 2\beta & \cos 2\beta \\ 1 & \sin 2\gamma & \cos 2\gamma \end{vmatrix} = 2 \begin{vmatrix} \sin(\alpha+\beta) & \cos(\alpha+\beta) & \cos(\alpha-\beta) \\ \sin(\beta+\gamma) & \cos(\beta+\gamma) & \cos(\beta-\gamma) \\ \sin(\gamma+\alpha) & \cos(\gamma+\alpha) & \cos(\gamma-\alpha) \end{vmatrix}.$$

2.27
$$\begin{vmatrix} 1 & a & a^2 & a^3 \\ a^3 & 1 & a & a^2 \\ x & a^3 & 1 & a \\ y & z & a^3 & 1 \end{vmatrix} = (1-a^4)^3.$$

Note that the result is independent of the values of x, y, z.

2.28
$$\begin{vmatrix} x & -1 & 0 & 0 \\ 0 & x & -1 & 0 \\ 0 & 0 & x & -1 \\ a & b & c & d+x \end{vmatrix} = \begin{vmatrix} x^2 & -1 & 0 \\ 0 & x & -1 \\ a+bx & c & d+x \end{vmatrix}$$

$$= \begin{vmatrix} x & -1 & 0 \\ 0 & x^2 & -1 \\ a & b+cx & d+x \end{vmatrix} = a+bx+cx^2+dx^3+x^4.$$

2.29
$$\begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix} = x^2-1; \qquad \begin{vmatrix} x & 1 & 0 \\ 2 & x & 2 \\ 0 & 1 & x \end{vmatrix} = x(x^2-2^2);$$

$$\begin{vmatrix} x & 1 & 0 & 0 \\ 3 & x & 2 & 0 \\ 0 & 2 & x & 3 \\ 0 & 0 & 1 & x \end{vmatrix} = (x^2-1)(x^2-3^2);$$

$$\begin{vmatrix} x & 1 & 0 & 0 & 0 \\ 4 & x & 2 & 0 & 0 \\ 0 & 3 & x & 3 & 0 \\ 0 & 0 & 2 & x & 4 \\ 0 & 0 & 0 & 1 & x \end{vmatrix} = x(x^2-2^2)(x^2-4^2).$$

2.30
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \frac{1}{\lambda^2} \begin{vmatrix} a_1 & b_1 & c_1 \\ \lambda a_2 - \mu a_1 & \lambda b_2 - \mu b_1 & \lambda c_2 - \mu c_1 \\ \lambda a_3 - \nu a_1 & \lambda b_3 - \nu b_1 & \lambda c_3 - \nu c_1 \end{vmatrix}, \quad \lambda \neq 0.$$

2.31
$$(a_1 b_2 c_3) = \frac{1}{a_1} \begin{vmatrix} (a_1 b_2) & (a_1 c_2) \\ (a_1 b_3) & (a_1 c_3) \end{vmatrix}, \quad a_1 \neq 0,$$

where

$$(a_1 b_2) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad (a_1 b_2 c_3) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

HINT. In **2.30** put $\lambda = a_1$, $\mu = a_2$, $\nu = a_3$.

2.32
$$(a_1 b_2 c_3 d_4) = \frac{1}{{a_1}^2} \begin{vmatrix} (a_1 b_2) & (a_1 c_2) & (a_1 d_2) \\ (a_1 b_3) & (a_1 c_3) & (a_1 d_3) \\ (a_1 b_4) & (a_1 c_4) & (a_1 d_4) \end{vmatrix}, \quad a_1 \neq 0,$$

$$= \frac{1}{(a_1 b_2)} \begin{vmatrix} (a_1 b_2 c_3) & (a_1 b_2 d_3) \\ (a_1 b_2 c_4) & (a_1 b_2 d_4) \end{vmatrix}, \quad (a_1 b_2) \neq 0.$$

2.33
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdot & \cdot & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdot & \cdot & a_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdot & \cdot & a_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & -a_{12} & -a_{13} & \cdot & \cdot & -a_{1n} \\ 0 & a_{11} & 0 & \cdot & \cdot & 0 \\ 0 & 0 & a_{11} & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & a_{11} \end{bmatrix}$$

$$= \begin{bmatrix} {a_{11}}^2 & 0 & 0 & & 0 \\ a_{11}a_{21} & (a_{11}a_{22}) & (a_{11}a_{23}) & & (a_{11}a_{2n}) \\ a_{11}a_{31} & (a_{11}a_{32}) & (a_{11}a_{33}) & & (a_{11}a_{3n}) \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ a_{11}a_{n1} & (a_{11}a_{n2}) & (a_{11}a_{n3}) & & (a_{11}a_{nn}) \end{bmatrix},$$

and hence

$$\begin{vmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{vmatrix} = {a_{11}}^{2-n} \begin{vmatrix} (a_{11}a_{22}) & (a_{11}a_{23}) & \cdot & \cdot & (a_{11}a_{2n}) \\ (a_{11}a_{32}) & (a_{11}a_{33}) & \cdot & \cdot & (a_{11}a_{3n}) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (a_{11}a_{n2}) & (a_{11}a_{n3}) & \cdot & \cdot & (a_{11}a_{nn}) \end{vmatrix},$$

provided that $a_{11} \neq 0$.

2.34
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \frac{1}{b_1} \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & (b_1 c_2) \\ a_3 & b_3 & (b_1 c_3) \end{vmatrix}$$
$$= \frac{1}{a_1 b_1} \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & (a_1 b_2) & (b_1 c_2) \\ a_3 & (a_1 b_3) & (b_1 c_3) \end{vmatrix},$$
$$a_1 b_1 \neq 0,$$

and, in general,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & . & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & . & . & a_{2n} \\ a_{31} & a_{32} & a_{33} & . & . & a_{3n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & a_{n3} & . & . & a_{nn} \end{vmatrix}$$
$$= \prod_{k=2}^{n-1} (a_{1k})^{-1} \begin{vmatrix} (a_{11}a_{22}) & (a_{12}a_{23}) & & (a_{1n-1}a_{2n}) \\ (a_{11}a_{32}) & (a_{12}a_{33}) & & (a_{1n-1}a_{3n}) \\ . & . & . & . \\ . & . & . & . \\ (a_{11}a_{n2}) & (a_{12}a_{n3}) & & (a_{1n-1}a_{nn}) \end{vmatrix},$$

provided that $\prod_{2}^{n-1} a_{1k} \neq 0$.

§ 3. Inverses

3.1 Prove that the inverse of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad (ad - bc \neq 0)$$

is

$$A^{-1} = \frac{1}{ad-bc} B, \qquad \text{where } B = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

It is useful to note that B is obtained from A by interchanging the diagonal elements and changing the sign of the off-diagonal elements.

3.2 Find all matrices A of order 2 with complex elements such that $A+\bar{A}$ and $A\bar{A}$ are scalar matrices. If $|A| \neq 0$, show that A^{-1} is of the same form.

HINT.

$A+\bar{A}$ = a scalar matrix implies $A = aI+iB$, a, B real; and $A-\bar{A}$ = a scalar matrix implies B^2 = a scalar matrix.

Hence

$$B = \begin{bmatrix} b & c \\ d & -b \end{bmatrix} \text{ and } A = \begin{bmatrix} a+ib & ic \\ id & a-ib \end{bmatrix}.$$

3.3 If

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 1 \\ 5 & 3 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 2 & -3 \\ -2 & 3 & 1 \\ 2 & -3 & -1 \end{bmatrix},$$

(a) find A^{-1} and B^{-1};

(b) express x in terms of z given that $y = Ax$ and $z = By$.

3.4 If

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \qquad \cos\tfrac{1}{2}\theta \neq 0,$$

prove that $I+A$ is non-singular and that

$$(I-A)(I+A)^{-1} = \tan(\tfrac{1}{2}\theta)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

3.5 If

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

solve the matrix equations
(a) $AX = B$, (b) $XA = B$, (c) $AX = BA^{-1}B$.

3.6 If

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 1 & 0 & 1 \end{bmatrix},$$

solve the matrix equations
(a) $(2I-A)X = I+A$, (b) $X(2I-A) = I+A$.

3.7 Prove that the set of matrices of the form

$$A(x) = \begin{bmatrix} 1 & 0 & x \\ -x & 1 & -\frac{1}{2}x^2 \\ 0 & 0 & 1 \end{bmatrix}$$

form a commutative group under matrix multiplication.

HINT. Prove that

$$A(x_1)A(x_2) = A(x_1+x_2).$$

3.8 If A, B, C, F, G, H are matrices of order n and A, B, C are non-singular, verify that

$$\begin{bmatrix} A & H & G \\ O & B & F \\ O & O & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}HB^{-1} & A^{-1}HB^{-1}FC^{-1}-A^{-1}GC^{-1} \\ O & B^{-1} & -B^{-1}FC^{-1} \\ O & O & C^{-1} \end{bmatrix}.$$

3.9 Find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

and check that $(A^{-1})^2 = (A^2)^{-1}$.

3.10 Verify that

(a) $$\begin{bmatrix} 2 & 3 & 1 & 2 \\ 2 & 1 & 5 & 0 \\ 3 & 0 & 2 & 3 \\ 1 & 4 & 0 & 3 \end{bmatrix}^{-1} = \frac{1}{64}\begin{bmatrix} 66 & -14 & 2 & -46 \\ 18 & 2 & -14 & 2 \\ -30 & 18 & 2 & 18 \\ -46 & 2 & 18 & 34 \end{bmatrix};$$

(b) $$\begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -a & 0 \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & a & a^2 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

and establish a result analogous to (b) for a matrix of order n.

3.11 Verify that

$$\begin{bmatrix} \cos\alpha & \sin\alpha & \cos\beta & \sin\beta \\ -\sin\alpha & \cos\alpha & -\sin\beta & \cos\beta \\ 0 & 0 & \cos\gamma & \sin\gamma \\ 0 & 0 & -\sin\gamma & \cos\gamma \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha & -\cos(\alpha-\beta+\gamma) & \sin(\alpha-\beta+\gamma) \\ \sin\alpha & \cos\alpha & -\sin(\alpha-\beta+\gamma) & -\cos(\alpha-\beta+\gamma) \\ 0 & 0 & \cos\gamma & -\sin\gamma \\ 0 & 0 & \sin\gamma & \cos\gamma \end{bmatrix}.$$

3.12 Find the inverses of the matrices

(a) $$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{(b)} \quad B = \begin{bmatrix} I_m & C \\ O & I_n \end{bmatrix},$$

where C is an $m \times n$ matrix.

3.13 Verify that

$$\begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ -1 & 0 & 1 & \cdots & 1 \\ -1 & -1 & 0 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & -1 & \cdots & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 & 1 & \cdots & -1 \\ 1 & 0 & -1 & \cdots & 1 \\ -1 & 1 & 0 & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & -1 & 1 & \cdots & 0 \end{bmatrix},$$

each matrix being a skew-symmetric matrix of even order.

3.14 If $a > 1$, prove that the matrix

$$\begin{bmatrix} a & 1 & 1 & \cdots & 1 \\ 1 & a & 1 & \cdots & 1 \\ 1 & 1 & a & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & a \end{bmatrix}$$

of order n is non-singular and determine its inverse.

3.15 The elements a_{ij} of a non-singular matrix A of even order are such that $a_{ij} = 0$ when $i+j$ is even. Prove that A^{-1} has the same form.

3.16 If A, B are matrices of order n and $I+A+AB = O$, prove that A is non-singular and that $A^{-1} = -I-B$.

3.17 If A is a square matrix and $A^k = O$, where k is a positive integer, prove that

$$(I-A)^{-1} = I+A+A^2+ \ldots +A^{k-1}.$$

HINT. Use the identity

$$I-A^n = (I-A)(I+A+A^2+ \ldots +A^{n-1}),$$

valid for any positive integer n.

3.18 If the matrix A satisfies the relation

$$I+A+A^2+ \ldots +A^k = O,$$

prove that $A^{-1} = A^k$.

HINT. Multiplication of the relation by A gives

$$A+A^2+A^3+\ldots+A^{k+1}=O,$$

so that $A^{k+1}=I$, from which the required result follows.

3.19 If A is a symmetric matrix and S a skew-symmetric matrix, each of order n, and $A+S$ is non-singular, prove that

$$T'(A+S)T=A+S, \qquad T'(A-S)T=A-S,$$

where $T=(A+S)^{-1}(A-S)$.

3.20 If S, T are matrices of the same order and $I+S$, $I+ST$, $I+TS$, $I-S$ are non-singular, prove that

$$(I+S)^{-1}(S+T)(I+ST)^{-1}(I+S) = (I-S)(I+TS)^{-1}(S+T)(I-S)^{-1}.$$

3.21 If A is a square matrix, λ, μ are scalars, and

$$A_\lambda=(\lambda I-A)^{-1}, \qquad A_\mu=(\mu I-A)^{-1},$$

prove that

$$(\lambda-\mu)A_\lambda A_\mu=A_\mu-A_\lambda.$$

HINT. Use the relation

$$A_\lambda^{-1}(A_\mu-A_\lambda)A_\mu^{-1}=A_\lambda^{-1}-A_\mu^{-1}.$$

3.22 If the sum of the elements in each column of a non-singular matrix is equal to a constant a, prove that the sum of the elements in each column of the inverse of the matrix is a^{-1}.

HINT. Let A be the matrix and e' the row $[1\ 1\ \ldots\ 1]$. Then

$$e'A=ae',$$

and multiplication on the right of each side of this equation by A^{-1} gives

$$e'=ae'A^{-1} \quad \text{or} \quad e'A^{-1}=a^{-1}e',$$

which is the required result.

NOTE. A similar result is true for a matrix in which the sum of the elements in each row is equal to a constant b. Then

$$Ae=be,$$

giving

$$A^{-1}e=b^{-1}e.$$

3.23 In the notation of the previous problem, $ee' = E$ is a square matrix with each element unity. If A and B are two non-singular matrices such that

$$A+B = kE, \qquad k \text{ a scalar},$$

prove that

$$\{1-kS(A^{-1})\}\{1-kS(B^{-1})\} = 1,$$

where for any matrix $C = [c_{ij}]$,

$$S(C) = \sum_{i,j} c_{ij}.$$

HINT. From the definition,

$$S(C) = e'Ce.$$

Since

$$A+B = kee',$$
$$I+A^{-1}B = kA^{-1}ee',$$

and hence $B^{-1}+A^{-1} = kA^{-1}ee'B^{-1}$ by pre- and post-multiplication by A^{-1} and B^{-1}, respectively. Hence

$$e'B^{-1}e+e'A^{-1}e = ke'A^{-1}ee'B^{-1}e,$$

or

$$S(B^{-1})+S(A^{-1}) = kS(A^{-1})S(B^{-1}),$$

from which the required result follows immediately.

3.24 Prove that, if A is a matrix of order n, then

(a) $\quad |\text{adj } A| = |A|^{n-1}$,

(b) $\quad \text{adj}(\text{adj } A) = |A|^{n-2}A$,

and hence determine the matrix A, if

$$\text{adj } A = \begin{bmatrix} 2 & -2 & 0 \\ -6 & 9 & -1 \\ 8 & -12 & 2 \end{bmatrix}.$$

3.25 If A and B are non-singular matrices which commute, prove that

(a) the following matrices commute:
A and B^{-1}, A^{-1} and B, A^{-1} and B^{-1}, A' and B';

(b) the matrix equations $AX = B$ and $YA = B$ have the same solution.

§ 4. Special Matrices

4.1 If

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & a_1 \\ 0 & -1 & a_1 & a_2 \\ -1 & a_1 & a_2 & a_3 \end{bmatrix}, \qquad B = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

prove that AB is symmetric and that $ABA^{-1} = B'$.

4.2 If A is an $n \times n$ skew-symmetric matrix and B is the $3 \times n$ matrix

$$B = \begin{bmatrix} b_1 & b_1 & . & b_1 \\ b_2 & b_2 & . & b_2 \\ b_3 & b_3 & . & b_3 \end{bmatrix},$$

find the matrix BAB'.

4.3 Given the matrices

$$A = \begin{bmatrix} 1 & i & 1+i \\ -i & 2 & 1 \\ 1-i & 1 & 3 \end{bmatrix}, \qquad B = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix},$$

$$C = \begin{bmatrix} -2i & 4+2i \\ -2+4i & 1+i \end{bmatrix},$$

prove that A is hermitian, B is unitary, and C is normal.

4.4 Verify that the following matrices are orthogonal:

(a) $\frac{1}{3} \begin{bmatrix} 2 & -2 & 1 \\ -2 & -1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$;

(b) $\frac{1}{a^2+ab+b^2} \begin{bmatrix} -ab & b(a+b) & a(a+b) \\ -b(a+b) & -a(a+b) & ab \\ a(a+b) & -ab & b(a+b) \end{bmatrix}$;

(c) $$\frac{1}{a^2+b^2+c^2+d^2}\begin{bmatrix} a^2+b^2-c^2-d^2 & 2(ac+bd) & 2(ad-bc) \\ 2(ac-bd) & b^2+c^2-a^2-d^2 & 2(ab+cd) \\ 2(ad+bc) & 2(cd-ab) & b^2+d^2-a^2-c^2 \end{bmatrix}.$$

NOTE. (c) If a, b, c, d are integers, this matrix is the most general orthogonal matrix of order 3 with rational numbers as elements.

4.5 If

$$l_1^2+m_1^2+n_1^2 = l_2^2+m_2^2+n_2^2 = l_3^2+m_3^2+n_3^2 = 1,$$
$$l_1l_2+m_1m_2+n_1n_2 = l_2l_3+m_2m_3+n_2n_3 = l_3l_1+m_3m_1+n_3n_1 = 0,$$

prove that

$$l_1^2+l_2^2+l_3^2 = m_1^2+m_2^2+m_3^2 = n_1^2+n_2^2+n_3^2 = 1,$$
$$l_1m_1+l_2m_2+l_3m_3 = m_1n_1+m_2n_2+m_3n_3 = n_1l_1+n_2l_2+n_3l_3 = 0.$$

HINT. The given relations imply that the matrix

$$A = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$$

satisfies the equation $AA' = I$, so that A is orthogonal. Hence $A'A = I$, from which the required relations follow.

4.6 If

$$A = \begin{bmatrix} 1 & c & -b \\ -c & 1 & a \\ b & -a & 1 \end{bmatrix}$$

and $x = Az$, $y = A'z$, prove that the transformation from x to y is an orthogonal transformation.

4.7 By evaluating the leading principal minors of the real symmetric matrix

$$A = \begin{bmatrix} 9 & -6 & 2 \\ -6 & 8 & -4 \\ 2 & -4 & 4 \end{bmatrix},$$

prove that this matrix is positive definite.

4.8 Prove that

$$A = \begin{bmatrix} 2 & 1+2i & 2-3i \\ 1-2i & 6 & -4-2i \\ 2+3i & -4+2i & 14 \end{bmatrix}$$

is a positive definite hermitian matrix.

4.9 If

$a = \sin\alpha \sin\beta$, $b = \sin\alpha \cos\beta$, $c = \cos\alpha \sin\beta$, $d = \cos\alpha \cos\beta$,

prove that

$$A = \begin{bmatrix} a & b & d & -c \\ b & -a & c & d \\ c & d & -b & a \\ d & -c & -a & -b \end{bmatrix}$$

is orthogonal.

4.10 Find positive numbers a, b, c, d, e, f, g, h, so that the matrix

$$A = \frac{1}{\sqrt{h}} \begin{bmatrix} a & \sqrt{f} & \sqrt{3} & c\sqrt{g} & \sqrt{f} \\ b & \sqrt{f} & -\sqrt{3} & 0 & -\sqrt{f} \\ c & 0 & c\sqrt{3} & -c\sqrt{g} & 0 \\ d & -\sqrt{f} & -\sqrt{3} & 0 & \sqrt{f} \\ e & -\sqrt{f} & \sqrt{3} & c\sqrt{g} & -\sqrt{f} \end{bmatrix}$$

is orthogonal.

4.11 Given that $I+A$ is a non-singular matrix, prove that
(a) if A is skew-symmetric, $(I-A)(I+A)^{-1}$ is orthogonal;
(b) if A is orthogonal, $(I-A)(I+A)^{-1}$ is skew-symmetric.

HINT. (a) If $B = (I-A)(I+A)^{-1}$, then

$$\begin{aligned} B'B &= (I+A')^{-1}(I-A')(I-A)(I+A)^{-1} \\ &= (I-A)^{-1}(I+A)(I-A)(I+A)^{-1} \\ &= (I-A)^{-1}(I-A)(I+A)(I+A)^{-1} \\ &= I. \end{aligned}$$

4.12 Prove that
(a) if A is idempotent, $2A-I$ is involutory;
(b) if A is involutory, $\frac{1}{2}(A+I)$ is idempotent.

4.13 Prove that if a matrix possesses two of the three properties: orthogonal, symmetric, involutory, it possesses the third.

4.14 If $A = X+iY$, where X and Y are real matrices, and $A^{-1} = \bar{A}$, prove that

$$X^2+Y^2 = I \quad \text{and} \quad XY = YX.$$

4.15 If A is a real symmetric matrix, prove that $I+\varepsilon A$ is positive definite if ε is sufficiently small.

4.16 A centro-symmetric matrix $A = [a_{ij}]$ of order n is defined by

$$a_{ij} = a_{n+1-i,\,n+1-j}, \; i, \quad j = 1, 2, \ldots, n.$$

For example

$$A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ h & g & f & e \\ d & c & b & a \end{bmatrix}$$

is a centro-symmetric matrix of order 4.

A matrix $K = [k_{ij}]$ of order n is defined by

$$k_{ij} = \begin{cases} 1, & i+j = n+1 \\ 0, & \text{otherwise.} \end{cases}$$

For example, when $n = 4$,

$$K = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Prove, for any n, that

(i) $K^2 = I =$ unit matrix of order n;
(ii) $(I+K)(I-K) = O$;
(iii) $KA = AK$;
(iv) $(I+K)A(I-K) = O$.

Verify (i), (ii), (iii) when $n = 4$.

(b) A centro-symmetric matrix B of order $2n$ is partitioned so that

$$B = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$$

where P, Q, R, S are matrices of order n. Prove that

(i) $KP = SK, \quad KQ = RK;$

(ii) $\begin{bmatrix} I & O \\ K & I \end{bmatrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \begin{bmatrix} I & O \\ -K & I \end{bmatrix} = \begin{bmatrix} P-QK & Q \\ O & S+KQ \end{bmatrix};$

(iii) $|B| = |P-QK|\,|P+QK|.$

HINT. (a) (i) $K^2 = [\sum_{p=1}^{n} k_{ip}k_{pj}]$

$$= [k_{i\,n+1-i}\,k_{n+1-i\,j}]$$
$$= [k_{n+1-i\,j}]$$
$$= [\delta_{ij}] = I.$$

(ii) $(I+K)(I-K) = I+K-K-K^2 = O.$

(b) (i) From (a) (iii) it follows that

$$\begin{bmatrix} O & K \\ K & O \end{bmatrix} \text{ commutes with } \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$$

so that

$$\begin{bmatrix} KR & KS \\ KP & KQ \end{bmatrix} = \begin{bmatrix} QK & PK \\ SK & RK \end{bmatrix},$$

from which the required relations follow.

(iii) Note that $|S+KQ| = |K|\,|S+KQ|\,|K|$

$$= |KSK+QK|$$
$$= |P+QK|.$$

4.17 A continuant matrix is a square matrix with all elements zero except those on the leading diagonal and the diagonals below and above this, e.g.,

$$A_n = \begin{bmatrix} p_1 & q_2 & 0 & . & 0 & 0 \\ r_2 & p_2 & q_3 & . & 0 & 0 \\ 0 & r_3 & p_3 & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & p_{n-1} & q_n \\ 0 & 0 & 0 & . & r_n & p_n \end{bmatrix}$$

is a continuant matrix of order n.

(a) Prove that $|A_n| = p_n|A_{n-1}| - q_n r_n|A_{n-2}|$.

(b) If $r_2 = r_3 = \ldots = r_n = -1$, prove that

$$\cfrac{1}{p_1 + \cfrac{q_2}{p_2 + \cfrac{q_3}{p_3 + \ldots + \cfrac{q_n}{p_n}}}},$$

can be written in the form x_n/y_n, where $y_n = |A_n|$ and $x_n =$ cofactor of p_1 in $|A_n|$.

HINT. (b) Use induction.

§ 5. Rank, Linear Forms and Equations

5.1. Find the ranks of the following matrices:

(a) $$A = \begin{bmatrix} 0 & 1 & 8 \\ 1 & -3 & 6 \\ 1 & -1 & 23 \end{bmatrix},$$

(b) $$A = \begin{bmatrix} 3 & -1 & 2 & 1 \\ 1 & 4 & 6 & 1 \\ 7 & -11 & -6 & 2 \end{bmatrix},$$

(c) $$A = \begin{bmatrix} -1 & 0 & 2 & 5 & 1 \\ 0 & -1 & 3 & 1 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 1 & 1 & -1 & 0 & 4 \end{bmatrix}.$$

5.2 Prove that the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$$

is 2 and express the third and fourth columns of the matrix as linear combinations of the first two columns.

5.3 If $a \neq b \neq c \neq a$ and $\alpha \neq \beta \neq \gamma \neq \alpha$, prove that the rank of the matrix

$$\begin{bmatrix} 1 & a & \alpha & a\alpha \\ 1 & b & \beta & b\beta \\ 1 & c & \gamma & c\gamma \end{bmatrix}$$

is equal to 3.

5.4 Determine x, so that the rank of each of the following matrices is equal to 3:

(a) $\begin{bmatrix} 6 & 3 & 5 & 9 \\ 5 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 2 & 1 & 1 & x \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ x & 2 & 2 & 2 \\ 9 & 9 & x & 3 \end{bmatrix}$.

ANSWER. (a) $x = 3$ (b) $x = 2$ or -6.

5.5 If $a, b, c, \ldots, l$ are positive constants, prove that the rank of the matrix

$$\begin{bmatrix} a & -d & g & j \\ b & e & -h & k \\ c & f & i & -l \end{bmatrix}$$

is equal to 3.

HINT. The three rows can be shown to be linearly independent. In fact, suppose that

$$\begin{aligned} pa+qb+rc &= 0, \\ -pd+qe+rf &= 0, \\ pg-qh+ri &= 0, \\ pj+qk-rl &= 0, \end{aligned}$$

where, without loss of generality, it can be assumed that $p > 0$. The four possibilities

$$\begin{aligned} q \geqq 0, &\quad r \geqq 0, \\ q \leqq 0, &\quad r \leqq 0, \\ q \leqq 0, &\quad r \geqq 0, \\ q \geqq 0, &\quad r \leqq 0, \end{aligned}$$

are excluded by considering each of the four equations in turn.

5.6 Determine x so that the rank of the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 & 4 \\ x & 4 & 10 & 1 \\ 1 & 7 & 17 & 3 \\ 2 & 2 & 4 & 3 \end{bmatrix}$$

is a minimum.

5.7 A matrix A of order $m \times n$ is partitioned so that

$$A = \begin{matrix} \\ r \\ m-r \end{matrix} \begin{matrix} \begin{matrix} r & n-r \end{matrix} \\ \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \end{matrix}.$$

If $|P| \neq 0$, prove that the rank of A is equal to r, if and only if

$$S = RP^{-1}Q.$$

HINT. Consider the identity

$$\begin{matrix} \\ r \\ m-r \end{matrix} \begin{matrix} \begin{matrix} r & m-r \end{matrix} \\ \begin{bmatrix} P^{-1} & O \\ -RP^{-1} & I_{m-r} \end{bmatrix} \end{matrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} I_r & P^{-1}Q \\ O & S-RP^{-1}Q \end{bmatrix},$$

or

$$CA = B.$$

Since C is non-singular, rank A = rank B; but the rank of B is equal to r, if and only if $S-RP^{-1}Q = O$.

5.8 Find the value of a so that the linear forms

$$\begin{aligned} f_1 &= 5x-2y+9z, \\ f_2 &= -x+y+3z, \\ f_3 &= x-ay+2az, \end{aligned}$$

are linearly dependent and find the linear relation between them.

5.9 Find the relation between a, b, c, d, given that the following forms are linearly dependent:

$$\begin{aligned} f_1 &= (1-a)x+y+z+w \\ f_2 &= x+(1-b)y+z+w \\ f_3 &= x+y+(1-c)z+w \\ f_4 &= x+y+z+(1-d)w. \end{aligned}$$

5.10 Given the linear forms

$$\begin{aligned} f_1 &= x_1+2x_2+x_3-ax_4+2x_5, \\ f_2 &= 2x_1+x_2+x_3+x_4-ax_5, \\ f_3 &= x_1+x_2+2x_3+2x_4-2x_5, \\ f_4 &= 2x_1+3x_2-5x_3-17x_4+bx_5, \end{aligned}$$

determine a, b such that f_4 is a linear combination of f_1, f_2, f_3, and find the linear relation.

ANSWER. $a = 3$, $b = 10$, $f_4 = 3f_1+2f_2-5f_3$.

5.11 Solve the set of equations

$$\begin{aligned} x+2y-3z+4w &= 9,\\ x \quad -z+w &= 1,\\ 3x-y+z &= -1,\\ -x+y \quad +2w &= 9,\\ 3x+y \quad +3w &= 9. \end{aligned}$$

5.12 Find a non-trivial solution of the equations

$$\begin{aligned} x+y-2z &= 0,\\ 2x+y-3z &= 0,\\ 2x-y-z &= 0,\\ 6x-y-5z &= 0,\\ 7x-3y-4z &= 0. \end{aligned}$$

5.13 If the equations

$$\begin{aligned} x &= by+cz+dw,\\ y &= ax+cz+dw,\\ z &= ax+by+dw,\\ w &= ax+by+cz, \end{aligned}$$

have a non-trivial solution and a, b, c, $d \neq -1$, prove that

$$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}+\frac{d}{1+d}=1.$$

5.14 If

$$\begin{aligned} f_1 &= 2x+3y-z-w,\\ f_2 &= x-y-2z-4w,\\ f_3 &= 3x+y+3z-2w,\\ f_4 &= 6x+3y \quad -7w, \end{aligned}$$

and $f_1 = f_2 = f_3 = f_4 = 0$ has a non-trivial solution for x, y, z, w, find a linear relation between f_1, f_2, f_3, f_4.

HINT. Let

$$\alpha f_1+\beta f_2+\gamma f_3+\delta f_4 = 0,$$

so that

$$\begin{bmatrix} 2 & 1 & 3 & 6 \\ 3 & -1 & 1 & 3 \\ -1 & -2 & 3 & 0 \\ -1 & -4 & -2 & -7 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The solution for α, β, γ, δ gives the coefficients in the required relation.

5.15 Find x_1 and x_2 if

$$\begin{aligned} x_1+x_2+x_3+x_4+x_5 &= 3, \\ x_1+2x_2+2x_3+3x_4+4x_5 &= 9, \\ 2x_1+x_2-2x_3+2x_4-3x_5 &= -16, \\ 3x_1+2x_2+3x_3+4x_4+x_5 &= 2, \\ -x_1+x_2-4x_3+4x_4+2x_5 &= -12. \end{aligned}$$

HINT. One approach is to partition the matrix:

$$\left[\begin{array}{cc|ccc} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 3 & 4 \\ \hline 2 & 1 & -2 & 2 & -3 \\ 3 & 2 & 3 & 4 & 1 \\ -1 & 1 & -4 & 4 & 2 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ \hline x_3 \\ x_4 \\ x_5 \end{array}\right] = \left[\begin{array}{c} 3 \\ 9 \\ \hline -16 \\ 2 \\ -12 \end{array}\right],$$

which is written as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix},$$

giving

$$\begin{aligned} A_{11}y_1+A_{12}y_2 &= a_1 \\ A_{21}y_1+A_{22}y_2 &= a_2. \end{aligned}$$

Elimination of y_2 from these equations gives

$$(A_{11}-A_{12}A_{22}^{-1}A_{21})y_1 = a_1-A_{12}A_{22}^{-1}a_2.$$

A convenient ordering of the computation is to calculate A_{22}^{-1}, then $A_{12}A_{22}^{-1}$, $A_{12}A_{22}^{-1}a_2$, $a_1-A_{12}A_{22}^{-1}a_2$, $A_{12}A_{22}^{-1}A_{21}$, $A_{11}-A_{12}A_{22}^{-1}A_{21}$. In this example, the final equation is

$$\begin{bmatrix} 67 & 63 \\ 101 & 105 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix},$$

so that $x_1 = -x_2 = 1$.

5.16 Given that

$$\begin{aligned} ax^3+bx^2+cx+d &= 0, \\ \alpha x^3+\beta x^2+\gamma x+\delta &= 0, \end{aligned}$$

have a common root, prove that it satisfies the equation

$$x\begin{vmatrix} a & c & d & 0 \\ 0 & b & c & d \\ \alpha & \gamma & \delta & 0 \\ 0 & \beta & \gamma & \delta \end{vmatrix} = -\begin{vmatrix} b & c & d & 0 \\ a & b & c & d \\ \beta & \gamma & \delta & 0 \\ \alpha & \beta & \gamma & \delta \end{vmatrix}.$$

HINT. Write the equations in the form

$$\begin{aligned} (ax+b)x^3+cx^2+dx &= 0, \\ ax^3+bx^2+cx+d &= 0, \\ (\alpha x+\beta)x^3+\gamma x^2+\delta x &= 0, \\ \alpha x^3+\beta x^2+\gamma x+\delta &= 0, \end{aligned}$$

giving

$$\begin{vmatrix} ax+b & c & d & 0 \\ a & b & c & d \\ \alpha x+\beta & \gamma & \delta & 0 \\ \alpha & \beta & \gamma & \delta \end{vmatrix} = 0,$$

from which the result follows.

5.17 If $xyz \neq 0$ and

$$\begin{aligned} (x+y)(x+z) &= bcyz, \\ (y+z)(y+x) &= cazx, \\ (z+x)(z+y) &= abxy, \end{aligned}$$

prove that

$$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} \begin{vmatrix} -a & 1 & 1 \\ 1 & -b & 1 \\ 1 & 1 & -c \end{vmatrix} = 0.$$

HINT. By multiplication of the three equations, show that

$$(x+y)(y+z)(z+x) = \pm abcxyz,$$

and hence that $y+z = \pm ax$.

5.18 Solve the equations

$$\begin{aligned} x+y+z &= 1, \\ ax+by+cz &= d, \\ a^2x+b^2y+c^2z &= d^2. \end{aligned}$$

5.19 If

$$\sum_{r=0}^{k} \binom{k+a}{r} x_{k-r} = b_k,$$

prove that

$$\sum_{\rho=0}^{k} (-1)^\rho \binom{k+a}{\rho} b_{k-\rho} = x_k,$$

where $k = 0, 1, 2, \ldots, n$ and a is a constant.

§ 6. Eigenvalues and Eigenvectors

6.1 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & a \\ \varepsilon & 1 \end{bmatrix},$$

and discuss their limiting values as $\varepsilon \to 0$.

HINT. Show that the eigenvalues are $1 \pm \sqrt{a\varepsilon}$.

6.2 Show that the matrices

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

have the same eigenvalues, but are not similar (i.e. there is no non-singular matrix P such that $P^{-1}A_1P = A_2$).

6.3 If

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 5 \end{bmatrix},$$

express the inverse of $B = 2A^4 - 12A^3 + 19A^2 - 29A + 37I$ in the form

$$B^{-1} = pA + qI.$$

HINT. From the Cayley-Hamilton Theorem show that

$$A^2 = 6A - 7I,$$

and hence express A^3, A^4 in terms of A, I, to obtain

$$B = A + 2I.$$

The constants p, q are determined from

$$(pA + qI)(A + 2I) = I.$$

6.4 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

6.5 Find a non-singular matrix P such that $P^{-1}AP$ is diagonal and

(a) $A = \begin{bmatrix} -10 & -6 \\ 18 & 11 \end{bmatrix}$ (b) $A = \begin{bmatrix} -1 & -4 & -4 \\ 1 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$.

HINT. (a) Prove that $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ are eigenvectors of A and take P as the matrix with these vectors as columns.

6.6 Find the eigenvalues of the matrices A and A', where

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}.$$

If P is the matrix the columns of which are the eigenvectors of A, verify that the rows of P^{-1} are eigenvectors of A'.

6.7 If the eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & b & c \\ 1 & b' & c' \\ 1 & b'' & c'' \end{bmatrix}$$

are $[1, 1, 1]'$, $[1, 0, -1]'$, $[1, -1, 0]'$, determine b, c, b', c', b'', c''.

6.8 Find the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the matrix

$$A = \begin{bmatrix} 5 & -2 & -1 \\ -1 & 6 & 1 \\ -1 & 2 & 5 \end{bmatrix},$$

and verify the Cayley–Hamilton theorem by showing that

$$A^3 - (\lambda_1+\lambda_2+\lambda_3)A^2 + (\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)A - \lambda_1\lambda_2\lambda_3 I = O.$$

6.9 Find the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and corresponding eigenvectors x_1, x_2, x_3 of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix},$$

and verify that

$$A = \sum_{i=1}^{3} \lambda_i \frac{x_i x'_i}{x'_i x_i}$$

HINT. Show, for example, that $\lambda_1 = 3$ and $x_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, giving

$$\lambda_1 \frac{x_1 x'_1}{x'_1 x_1} = \tfrac{3}{9} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} [2 \quad -1 \quad 2]$$

$$= \tfrac{1}{3} \begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}.$$

6.10 If a, b, c are real and $a^2+b^2+c^2 = 1$, find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

and verify that $A^3 = -A$. Hence prove formally that

$$\sum_{r=0}^{\infty} \frac{\theta^r}{r!} A^r = I + A \sin\theta + A^2(1-\cos\theta).$$

6.11 Find the characteristic polynomial of the n-square matrix A with non-zero elements

$$a_{ij} = 1, \quad i = 1, 2, \ldots, n-1; \quad j = i+1,$$
$$a_{nj} = -p_{n-j+1}, \quad j = 1, 2, \ldots, n.$$

6.12 The non-zero elements of a matrix A of order $2n-1$ are

$$a_{ii} = c, \qquad i = 1, 2, \ldots, 2n-1,$$
$$a_{jj+1} = a_{j+1j} = 1, \qquad j = 1, 2, \ldots, 2n-2.$$

Verify that the eigenvectors of A are

$$x_k = \left[\sin\frac{k\pi}{2n} \sin\frac{2k\pi}{2n} \ldots \sin\frac{(2n-1)k\pi}{2n}\right]', \quad k = 1, 2, \ldots, 2n-1,$$

and hence find the eigenvalues of A.

HINT. The matrix A is a *band*, or tridiagonal, matrix of the form

$$A = \begin{bmatrix} c & 1 & 0 & & 0 & 0 \\ 1 & c & 1 & & 0 & 0 \\ 0 & 1 & c & & 0 & 0 \\ . & . & . & & . & . \\ . & . & . & & . & . \\ 0 & 0 & 0 & & c & 1 \\ 0 & 0 & 0 & & 1 & c \end{bmatrix}.$$

Show directly that

$$Ax_k = \left(c+2\cos\frac{k\pi}{2n}\right)x_k.$$

6.13 The non-zero elements of a matrix A of order n are

$$a_{i\,i+1} = a_{n1} = 1, \qquad i = 1, 2, \ldots, n-1.$$

Find the matrices $A, A^2, \ldots, A^n$ and the eigenvalues and eigenvectors of the matrix A and the matrix

$$B = \begin{bmatrix} a_1 & a_2 & a_3 & . & & a_{n-1} & a_n \\ a_n & a_1 & a_2 & . & & a_{n-2} & a_{n-1} \\ a_{n-1} & a_n & a_1 & . & & a_{n-3} & a_{n-2} \\ . & . & . & . & & . & . \\ . & & & . & & & \\ . & . & . & . & . & . & . \\ a_2 & a_3 & a_4 & . & . & a_n & a_1 \end{bmatrix}.$$

HINT. Show that $A^n = I$ and that the eigenvalues of A are the n n-th roots of unity. The matrix B is called a *cyclic* matrix and can be written as a polynomial in A:

$$B = a_1 I + a_2 A + a_3 A^2 + \ldots + a_n A^{n-1}.$$

6.14 The non-zero elements of a matrix A of order n are

$$a_{i\,i+1} = -a_{n1} = 1, \qquad i = 1, 2, \ldots, n-1.$$

Find the eigenvalues and eigenvectors of the matrix A and the matrix

$$B = \begin{bmatrix} a_1 & a_2 & a_3 & . & a_{n-1} & a_n \\ -a_n & a_1 & a_2 & . & a_{n-2} & a_{n-1} \\ -a_{n-1} & -a_n & a_1 & . & a_{n-3} & a_{n-2} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ -a_2 & -a_3 & -a_4 & . & -a_n & a_1 \end{bmatrix}.$$

6.15 If a' is the $n \times 1$ column with elements $a_1, a_2, \ldots, a_n$, prove that the eigenvalues of the matrix aa' are

$$\lambda_1 = \sum_{j=1}^{n} a_j^2, \qquad \lambda_2 = \lambda_3 = \ldots = \lambda_n = 0.$$

6.16 The elements a_{ij} of a *stochastic* matrix A of order n satisfy the relations

$$0 \leqq a_{ij} < 1, \quad \sum_{j=1}^{n} a_{ij} = 1, \quad i = 1, 2, \ldots, n.$$

Prove that $\lambda = 1$ is an eigenvalue of A and that the elements of the matrix A^k, k a positive integer, satisfy the same relations as the a_{ij}.

§ 7. Elementary Canonical Forms

7.1 Prove the matrix equivalence relation $\sim$ possesses the following properties:.

reflexive: $A \sim A$ all A,
symmetric: $A \sim B$ implies $B \sim A$,
transitive: $A \sim B$ and $B \sim C$ implies $A \sim C$.

Prove also that congruence, hermitian congruence and similarity possess these properties.

7.2 If

$$A = \begin{bmatrix} 2 & -5 \\ -5 & 13 \end{bmatrix}, \qquad B = \begin{bmatrix} 5 & -11 \\ -11 & 25 \end{bmatrix},$$

find a real matrix P such that $P'AP = I$ and $P'BP$ is diagonal.

HINT. By direct computation,

$$A^{-1}B = \begin{bmatrix} 10 & -18 \\ 3 & -5 \end{bmatrix},$$

with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 4$; corresponding eigenvectors are $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and these are the columns of P.

7.3 Find the signature of the hermitian matrix

$$A = \begin{bmatrix} 1 & 1-2i & 2+3i \\ 1+2i & 5 & -4+2i \\ 2-3i & -4-2i & 13 \end{bmatrix}.$$

HINT. By elementary row and column operations show that A is hermitian congruent to

$$\bar{P}'AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

so that the signature of A is (2,1).

7.4 Find the real orthogonal transformation which reduces the matrix

$$A = \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}$$

to a diagonal matrix $D = P'AP$. Evaluate D^{100} and hence determine A^{100}.

HINT. Show that the normalised eigenvectors of A are

$$\tfrac{1}{3}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \quad \tfrac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \tfrac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix},$$

and take these as the columns of P. To obtain A^{100}, use the fact that

$$A^{100} = PP'A^{100}PP' = P(P'AP)^{100}P'.$$

7.5 Given the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 3 & 6 & 2 & 1 \\ 1 & 2 & 0 & 3 \end{bmatrix},$$

(a) show by means of a sequence of elementary row operations that

$$A \sim \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

(b) show by means of a sequence of elementary column operations that

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix};$$

(c) show by means of a sequence of elementary row and column operations that

$$A \sim B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In (c), find Q and P if $B = QAP$.

HINT. (a) Show that

$$A \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

7.6 (a) Express the matrix

$$A = \begin{bmatrix} 1 & 4 & 1 & 3 \\ 0 & -1 & 3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & -2 & 5 & 1 \end{bmatrix}$$

as the product of a lower triangular matrix L, and an upper triangular matrix U with diagonal elements 1.
(b) Determine matrices Q, P such that $QAP = I$.

HINT. (a) Let

$$A = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} & u_{14} \\ 0 & 1 & u_{23} & u_{24} \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and determine the unknowns successively in the order l_{11}, l_{21}, l_{31}, l_{41}, in the first column of L, u_{12}, u_{13}, u_{14} in the first row of U, the second column of L, the second row of U, and so on.

The answer is

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 3 & -11 & -36 & 0 \\ 1 & -6 & -14 & {}^{22}/_{9} \end{bmatrix} \begin{bmatrix} 1 & 4 & 1 & 3 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & {}^{-1}/_{9} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) Note that $Q = L^{-1}$ and $P = U^{-1}$.

7.7 Factorise the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & -3 & 2 & 2 \\ -1 & -3 & -2 & 2 \\ 3 & 1 & -1 & 0 \end{bmatrix}$$

in the form $A = LU$ and hence solve the equations

$$\begin{aligned} x+ y+ z- w &= 1 \\ 2x-3y+2z+2w &= -2 \\ -x-3y-2z+2w &= -1 \\ 3x+ y- z &= 4. \end{aligned}$$

HINT. The equations

$$LUx = b,$$

are solved in two stages:

find x_1 so that $Lx_1 = b$;
find x so that $Ux = x_1$.

The answer is $x = 1$, $y = 0$, $z = w = -1$.

7.8 Let A be a real orthogonal matrix of order $2n$ with eigenvalues $\exp(\pm i\theta_r)$, $r = 1, \ldots, n$, $\theta_r \neq 0, \pi$.
If $z_1, \bar{z}_1, \ldots, z_r, \bar{z}_r, \ldots, z_n, \bar{z}_n$ are the normalised eigenvectors of A and

$$w_{2r-1} = \frac{1}{\sqrt{2}}(z_r + \bar{z}_r), \quad w_{2r} = \frac{-i}{\sqrt{2}}(z_r - \bar{z}_r),$$

prove that $W = [w_1\, w_2\, \ldots\, w_{2n}]$ is a real orthogonal matrix. Evaluate $W'AW$.

HINT. Since A is real,

$$Az_1 = \exp(i\theta_1)z_1$$

implies

$$A\bar{z}_1 = \exp(-i\theta_1)\bar{z}_1,$$

and since

$$\bar{z}'_1 z_1 = z'_1 z_1 = 0 \quad \text{and} \quad \bar{z}'_1 z_1 = 1,$$

$$w'_1 w_1 = 1, \qquad w'_1 w_2 = 0, \text{ etc.},$$

making W a real orthogonal matrix. It can now be proved that

$$\begin{aligned} Aw_1 &= \cos\theta w_1 + \sin\theta w_2, \\ Aw_2 &= -\sin\theta w_1 + \cos\theta w_2, \end{aligned}$$

and hence that $\begin{bmatrix} w_1' \\ w_2' \end{bmatrix} [Aw_1 Aw_2] = D_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}$, and finally, that

$$W'AW = \begin{bmatrix} D_1 & O & & O \\ O & D_2 & & O \\ . & . & & \\ . & . & & . \\ O & O & & D_n \end{bmatrix},$$

where

$$D_r = \begin{bmatrix} \cos\theta_r & -\sin\theta_r \\ \sin\theta_r & \cos\theta_r \end{bmatrix}.$$

7.9 Let A be a real non-singular skew-symmetric matrix of order $2n$ with eigenvalues $\pm i\theta_r$, θ_r real, with normalised eigenvectors $z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n$.
If W is the real orthogonal matrix defined as in **7.8**, prove that

$$W'AW = \begin{bmatrix} E_1 & O & & O \\ O & E_2 & & O \\ . & . & & \\ . & . & & . \\ O & O & & E_n \end{bmatrix},$$

where

$$E_r = \begin{bmatrix} 0 & \theta_r \\ -\theta_r & 0 \end{bmatrix}.$$

7.10 The non-zero elements of a real n-square continuant matrix A are

$$\begin{aligned} a_{ii} &= p_i, & i &= 1, 2, \ldots, n, \\ a_{i\,i+1} &= q_{i+1}, & i &= 1, 2, \ldots, n-1, \\ a_{i+1\,i} &= r_{i+1}, & i &= 1, 2, \ldots, n-1. \end{aligned}$$

If $q_{i+1}\, r_{i+1} > 0$, $i = 1, 2, \ldots, n-1$, prove that A can be transformed by a similarity transformation with a diagonal matrix D to a matrix $D^{-1}AD$ which is real symmetric.

7.11 If A is a positive definite real symmetric matrix, prove that its largest eigenvalue λ is given by

$$\lambda = \operatorname*{Max}_{x \text{ real}} \frac{x'Ax}{x'x}.$$

HINT. Let P be the real orthogonal matrix such that

$$P'AP = \begin{bmatrix} \lambda_1 & 0 & . & 0 \\ 0 & \lambda_2 & . & 0 \\ . & . & . & . \\ . & . & . & . \\ 0 & 0 & . & \lambda_n \end{bmatrix},$$

where $\lambda_1 > \lambda_2 > \ldots > \lambda_n > 0$. Then, if $x = Py$,

$$\frac{x'Ax}{x'x} = \frac{\lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_n y_n^2}{y_1^2 + y_2^2 + \ldots + y_n^2} \leqq \lambda_1.$$

7.12 Prove that if A, B are real symmetric matrices of order n, and A is positive definite, then

(a) $$\int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_n \exp\left(-\tfrac{1}{2}x'Ax\right) = (2\pi)^{\frac{1}{2}n} |A|^{-\frac{1}{2}};$$

(b) $$\int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_n \exp\left(x'y - \tfrac{1}{2}x'Ax\right)$$
$$= (2\pi)^{\frac{1}{2}n} |A|^{-\frac{1}{2}} \exp\left(\tfrac{1}{2}y'A^{-1}y\right);$$

(c) $$\int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_n (x'Bx) \exp\left(-\tfrac{1}{2}x'Ax\right)$$
$$= (2\pi)^{\frac{1}{2}n} |A|^{-\frac{1}{2}} \operatorname{tr}(A^{-1}B).$$

HINT. (a) Let U be the non-singular upper triangular matrix such that $U'AU = I$. Note that, since

$$|U'AU| = 1, \qquad |U| = |A|^{-\frac{1}{2}}.$$

Let $x = Uy$; the Jacobian of this transformation is $|U| = |A|^{-\frac{1}{2}}$. The multiple integral becomes

$$\int_{-\infty}^{\infty} \mathrm{d}y_1 \ldots \int_{-\infty}^{\infty} \mathrm{d}y_n \exp(-\tfrac{1}{2}y'y)|A|^{-\frac{1}{2}} = (2\pi)^{\frac{1}{2}n}|A|^{-\frac{1}{2}},$$

using

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\,\mathrm{d}y = \sqrt{(2\pi)}.$$

Alternatively, the matrix A can be diagonalised by a real orthogonal transformation.

(b) Use the transformation $x = z + A^{-1}y$ and show that $x'y - \frac{1}{2}x'Ax = \frac{1}{2}y'A^{-1}y - \frac{1}{2}z'Az$, from which the result follows, using (a).

(c) Simultaneously reduce A to the unit matrix and B to the diagonal matrix with the eigenvalues of $A^{-1}B$ as diagonal elements.

7.13 Evaluate

$$J = \int_{-\infty}^{\infty} \mathrm{d}x_1 \ldots \int_{-\infty}^{\infty} \mathrm{d}x_n \int_{-\infty}^{\infty} \mathrm{d}y_1 \ldots \int_{-\infty}^{\infty} \mathrm{d}y_n \exp(-\tfrac{1}{2}\bar{z}'Az),$$

where $z = x + iy$ and A is a positive definite hermitian matrix.

HINT. Let $A = P + iQ$ when P is symmetric and Q skew-symmetric; then

$$\bar{z}'Az = x'Px - 2x'Qy + y'Py$$

and

$$\int_{-\infty}^{\infty} \mathrm{d}x_1 \ldots \int_{-\infty}^{\infty} \mathrm{d}x_n \exp(x'Qy - \tfrac{1}{2}x'Px)$$
$$= (2\pi)^{\frac{1}{2}n}|P|^{-\frac{1}{2}} \exp(-\tfrac{1}{2}y'QP^{-1}Qy).$$

Hence

$$J = (2\pi)^{\frac{1}{2}n}|P|^{-\frac{1}{2}}\int_{-\infty}^{\infty} dy_1 \ldots \int_{-\infty}^{\infty} dy_n \exp\left(-\tfrac{1}{2}y'Py - \tfrac{1}{2}y'QP^{-1}Qy\right)$$

$$= (2\pi)^n|P|^{-\frac{1}{2}}|P+QP^{-1}Q|^{-\frac{1}{2}}$$

$$= (2\pi)^n|P|^{-\frac{1}{2}}|P+iQ|^{-\frac{1}{2}}|I-iP^{-1}Q|^{-\frac{1}{2}}$$

$$= (2\pi)^n|P+iQ|^{-\frac{1}{2}}|P-iQ|^{-\frac{1}{2}}$$

$$= (2\pi)^n|A|^{-\frac{1}{2}}|A'|^{-\frac{1}{2}}$$

$$= (2\pi)^n|A|^{-1}.$$